第一章　行列式

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两个重要问题：线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

线性方程组　　线性系统

$$\underbrace{A = \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{\text{矩阵，表格？}}, \quad b = \overbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}^{\text{向量}}}_{\text{给定的数据}}, \quad \underbrace{x = \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}^{\text{向量}}}_{\text{未知量}}$$

两个重要问题：线性方程组（续）

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \ \cdots \ \cdots \ \cdots\cdots \ \cdots \ \cdots \ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

线性方程组　　线性系统

- 解的存在性、唯一性、解的结构；

寻求：　$(x_1, x_2, \cdots, x_n)$

- 应用极广：代数方程、微分方程、最优化，$\cdots$；
- 困难：大规模问题（m 与 n 非常大），病态问题，$\cdots$。

两个重要问题：特征问题

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = \lambda x_2 \\ \cdots\ \cdots\ \cdots\ \cdots\cdots\ \cdots\ \cdots\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n \end{array}\right.$$

特征问题

$$A = \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}^{\text{数据：方阵}}, \quad \overbrace{\lambda}^{\text{特征值}}, \quad 0 \neq x = \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}^{\text{特征向量}}$$

二元方程组的消元

$$\begin{aligned}(a_{22}) &\to a_{11}x_1 + a_{12}x_2 = b_1\\ (a_{12}) &\to a_{21}x_1 + a_{22}x_2 = b_2\end{aligned} \Rightarrow \begin{cases} a_{11}a_{22}x_1 + a_{12}a_{22}x_2 = a_{22}b_1\\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2\end{cases}$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \qquad \begin{pmatrix} a_{11}a_{22} & a_{12}a_{22} & a_{22}b_1\\ a_{12}a_{21} & a_{12}a_{22} & a_{12}b_2\end{pmatrix}$$

$$\begin{aligned}(a_{21}) &\to a_{11}x_1 + a_{12}x_2 = b_1\\ (a_{11}) &\to a_{21}x_1 + a_{22}x_2 = b_2\end{aligned} \Rightarrow \begin{cases} a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = a_{21}b_1\\ a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = a_{11}b_2\end{cases}$$

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \qquad \begin{pmatrix} a_{11}a_{22} & a_{12}a_{22} & a_{22}b_1\\ a_{12}a_{21} & a_{12}a_{22} & a_{12}b_2\end{pmatrix}$$

二阶行列式

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22} - a_{12}a_{21} \quad \text{二阶行列式}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} := a_{22}b_1 - a_{12}b_2, \quad x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{D_1}{D}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} := a_{11}b_2 - a_{21}b_1, \quad x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} = \frac{D_2}{D}$$

$$\underbrace{\begin{array}{l} \Pi_1: \ a_{11}x_1 + a_{12}x_2 = b_1 \\ \Pi_2: \ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}}_{\text{两条直线}} \qquad \underbrace{\begin{array}{l} \vec{n_1} = (a_{11}, a_{12}) \\ \vec{n_2} = (a_{21}, a_{22}) \end{array}}_{\text{法向量}}$$

$$\overbrace{|D| = |\vec{n_1} \times \vec{n_2}|}^{\text{平行四边形有向面积}}$$

$$D = 0 \Leftrightarrow \vec{n_1} \, // \, \vec{n_2}$$

三阶行列式

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ 三阶行列式

$$:= a_{11}a_{22}a_{33}+a_{13}a_{21}a_{32}+a_{12}a_{23}a_{31}$$
$$-a_{13}a_{22}a_{31}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})$$
$$+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$ 按第一行展开

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

三阶行列式（续）

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ 三阶行列式

$$\begin{aligned} := \quad & a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} \\ & -a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ = \quad & a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) \\ & + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \end{aligned}$$

按第一列展开

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} - a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

三阶行列式（续）

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ 三阶行列式

按第 2 行/列展开

$$= -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

按第 3 行/列展开

$$= a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

三阶行列式（续）

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(j_1, j_2, j_3) 是$(1,2,3)$的排列

$\tau(j_1, j_2, j_3)$ 是逆序数

$$:= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}$$
$$-a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$3! = 6$个乘积

$$= \sum_{j_1 j_2 j_3} (-1)^{\tau(j_1, j_2, j_3)} a_{1j_1} a_{2j_2} a_{3j_3}$$

每行每列取且只取一个元素

$\tau(j_1, j_2, j_3)$
$\|$
j_1后面比它小的指标个数
$+$
j_2后面比它小的指标个数

偶逆序数	奇逆序数
$\tau(1,2,3) = 0$	$\tau(3,2,1) = 3$
$\tau(3,1,2) = 2$	$\tau(1,3,2) = 1$
$\tau(2,3,1) = 2$	$\tau(2,1,3) = 1$

n 阶行列式定义

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m\times n}$$

矩阵，数表
$m \neq n$长方
$m = n$方阵

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|_{n\times n}$$

对应一个数值
$:= |A| = det(A)$
方阵才有行列式

$$\underbrace{D_1 = |a_{ij}|_1 := a_{11},}_{\text{不是绝对值/模}} \quad D_n = |a_{ij}|_{n\times n} := \underbrace{\sum_{j=1}^{n} a_{1j} \overbrace{(-1)^{1+j} M_{1j}}^{\text{代数余子式}}}_{\text{递归定义}}$$

n 阶行列式定义（续）

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = |a_{ij}|_{n\times n}$$

删去第i行
删去第j列 $\Rightarrow$

$$M_{ij} := \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

余子式
$(n-1)$阶

$$A_{ij} := (-1)^{i+j} M_{ij} \longleftarrow$$ 代数余子式

n 阶行列式定义（续，等价定义）

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j}$$

$$:= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1,j_2,\cdots,j_n)} \underbrace{a_{1j_1}a_{2j_2}\cdots a_{nj_n}}_{n!\text{个乘积}}$$

每行每列取且只取一个元素

$$\tau(j_1,j_2,\cdots j_n) = \sum_{k=1}^{n-1} j_k\text{后面比它小的指标个数}$$ ⟵ 逆序数

$$D_4 = \begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{vmatrix} := \underbrace{(-1)^{\overbrace{\tau(4,3,2,1)}^{3+2+1=6}}}_{=+1} a_{14}a_{23}a_{32}a_{41}.$$

n 阶行列式定义（续，等价定义）

$$D_4 = \begin{vmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{vmatrix} = (-1)^{1+4} a_{14} \times \begin{vmatrix} 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & 0 \end{vmatrix}$$

$$= (-1)^{1+4} a_{14} \times (-1)^{1+3} a_{23} \begin{vmatrix} 0 & a_{32} \\ a_{41} & 0 \end{vmatrix} \quad (|a_{41}| = a_{41})$$

$$= (-1)^{1+4} a_{14} \times (-1)^{1+3} a_{23} \times (-1)^{1+2} a_{32} \times |a_{41}|$$

$$= a_{14} a_{23} a_{32} a_{41}$$ 特殊结构的行列式，用定义可以算出

- 一般情况下，用定义计算很繁琐，需要发展一套计算方法，程式化，计算效率高。。。

n 阶行列式定义（续，下三角行列式）

$$D_n = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & 0 & \cdots & 0 \\ a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}a_{22}\cdots a_{k-1,k-1} \begin{vmatrix} a_{kk} & 0 & \cdots & 0 \\ a_{k+1,k} & a_{k+1,k+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{nk} & a_{n,k+1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}a_{22}\cdots a_{nn}$$

反复按第一行展开

对于上三角行列式，怎么计算？

行列式按第一列展开

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} := \sum_{j=1}^{n} a_{1j}(-1)^{1+j}M_{1j} = \sum_{i=1}^{n} a_{i1}(-1)^{i+1}M_{i1}?$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{=a_{11}M_{11}-a_{21}M_{21}}$$

归纳奠基✓

归纳假定：$(k = n-1)$

$$\underbrace{\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{vmatrix}}_{\text{任意}k=n-1\text{阶行列式}} = \sum_{j=1}^{n-1} b_{1j}(-1)^{1+j}M_{1j} = \sum_{i=1}^{n-1} b_{i1}(-1)^{i+1}M_{i1}$$

$k = n$阶？
归纳证明？

行列式按第一列展开（续）

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} := a_{11}M_{11} + \sum_{j=2}^{n} a_{1j}(-1)^{1+j}M_{1j}$$

$$M_{1j} = \begin{vmatrix} a_{21} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix}$$

删去D的第1行
删去D的第j列
得到M_{1j}，$(n-1)$阶

$$= \sum_{k=2}^{n} a_{k1}(-1)^{k}(M_{1j})_{k1}$$

$\leftarrow$ 删去D的第$1,k$行，第$1,j$列

由归纳假定，按第一列展开

$$D = a_{11}M_{11} + \sum_{j,k=2}^{n} a_{1j}a_{k1}(-1)^{1+j+k}(M_{1j})_{k1}$$

行列式按第一列展开（续）

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \underbrace{= a_{11}M_{11} + \sum_{k=2}^{n} a_{k1}(-1)^{k+1}M_{k1}?}_{=:C}$$

$$M_{k1} = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k-1,2} & a_{k-1,3} & \cdots & a_{k-1,n} \\ a_{k+1,2} & a_{k+1,3} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

删去D的第k行
删去D的第1列
得到M_{k1}，按第 1 行展开

$$= \sum_{j=2}^{n} a_{1j}(-1)^{j}(M_{k1})_{1j}$$

$$C = a_{11}M_{11} + \sum_{j,k=2}^{n} a_{k1}a_{1j}(-1)^{1+k+j}(M_{k1})_{1j}$$

$\leftarrow$ 删去D的第$1, k$行，第$1, j$列

行列式按第一列展开（续）

$$D = a_{11}M_{11} + \sum_{j,k=2}^{n} a_{1j}a_{k1}(-1)^{1+j+k}(M_{1j})_{k1}$$

$$C = a_{11}M_{11} + \sum_{j,k=2}^{n} a_{k1}a_{1j}(-1)^{1+k+j}(M_{k1})_{1j}$$

$$\Downarrow$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{1j} \overbrace{(-1)^{1+j}M_{1j}}^{=A_{1j}} \quad \text{定义}$$

$$= \sum_{i=1}^{n} a_{i1} \underbrace{(-1)^{i+1}M_{i1}}_{=A_{i1}} \quad \text{证明}$$

行列式按第一列展开（续，上三角行列式）

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}a_{22}\cdots a_{k-1,k-1} \begin{vmatrix} a_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ 0 & a_{k+1,k+1} & \cdots & a_{k+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11}a_{22}\cdots a_{nn}$$ 反复按第一列展开

将一般行列式等值变换为三角行列式？

行列式 = 转置行列式

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =: D_n = D'_n := \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} \quad ?$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$$ 归纳奠基(✓)

$$D = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j}$$ 按第 1 行展开

$\|$ (✓)

$$D' = \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M'_{1j}$$ 按第 1 列展开

$\|$ 归纳假定

$$D_{n-1} = D'_{n-1}$$
$$\Downarrow$$
$$M_{1j} = M'_{1j}$$

交换行列式行/列，变号！（对调 1、2 行）

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = D = -\begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} \quad ?$$

$$C = \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} a_{2j}(-1)^{1+j} \overbrace{M_{1j}^{c}}^{=M_{2j}} \quad \text{第 1 行展开}$$

$$= \sum_{j=1}^{n} a_{2j}(-1)^{1+j} M_{2j}$$

$$D = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} M_{1k}$$

展开 M_{2j}, M_{1k} 的第 1 行

交换行列式行/列，变号！（对调 1、2 行，续）

$$M_{2j} = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{vmatrix} \quad \underbrace{\sum_{k=1}^{n} a_{1k}(-1)^{1+k} M_{1k}}_{C=}$$

$$= \sum_{k=1}^{j-1} a_{1k}(-1)^{1+k}(M_{2j})_{1k} + \sum_{k=j+1}^{n} a_{1k}(-1)^{k}(M_{2j})_{1k}$$

$$C = \sum_{j=1}^{n} a_{2j}(-1)^{1+j} \sum_{k=1}^{j-1} a_{1k}(-1)^{1+k}(M_{2j})_{1k} \qquad k < j$$

$$+ \sum_{j=1}^{n} a_{2j}(-1)^{1+j} \sum_{k=j+1}^{n} a_{1k}(-1)^{k}(M_{2j})_{1k} \qquad k > j$$

交换行列式行/列，变号！（对调 1、2 行，续）

$$M_{1k} = \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{n,n} \end{vmatrix} \quad \underbrace{\sum_{k=1}^{n} a_{1k}(-1)^{1+k} M_{1k}}_{D=}$$

$$= \sum_{j=1}^{k-1} a_{2j}(-1)^{1+j}(M_{1k})_{2j} + \sum_{j=k+1}^{n} a_{2j}(-1)^{j}(M_{1k})_{2j}$$

$$D = \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \sum_{j=1}^{k-1} a_{2j}(-1)^{1+j}(M_{1k})_{2j} \qquad k > j$$

$$+ \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \sum_{j=k+1}^{n} a_{2j}(-1)^{j}(M_{1k})_{2j} \quad k < j$$

交换行列式行/列，变号！(对调 1、2 行，续)

$$
\begin{aligned}
C &= \sum_{j=1}^{n} a_{2j}(-1)^{1+j} \sum_{k=1}^{j-1} a_{1k}(-1)^{1+k}(M_{2j})_{1k} \\
&\quad + \sum_{j=1}^{n} a_{2j}(-1)^{1+j} \sum_{k=j+1}^{n} a_{1k}(-1)^{k}(M_{2j})_{1k} \\
&= \sum_{k<j} a_{1k}a_{2j}(-1)^{k+j}(M_{2j})_{1k} + \sum_{k>j} a_{1k}a_{2j}(-1)^{k+j+1}(M_{2j})_{1k} \\
D &= \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \sum_{j=1}^{k-1} a_{2j}(-1)^{1+j}(M_{1k})_{2j} \\
&\quad + \sum_{k=1}^{n} a_{1k}(-1)^{1+k} \sum_{j=k+1}^{n} a_{2j}(-1)^{j}(M_{1k})_{2j} \\
&= \sum_{j<k} a_{1k}a_{2j}(-1)^{k+j}(M_{1k})_{2j} + \sum_{j>k} a_{1k}a_{2j}(-1)^{k+j+1}(M_{1k})_{2j}
\end{aligned}
$$

交换行列式行/列，变号！（对调相邻两行）

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = -\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}$$ 归纳奠基(✓)

假定：对调$(n-1)$阶行列式相邻两行，行列式变号

- 对调 n 阶行列式第 1、2 行，行列式变号；归纳假定(✓)
- 对调 n 阶行列式第 i、$i+1$ 行 $(i>1)$，得到行列式 C，$C(1,:) = D(1,:)$

$$C\text{的}(1,j)\text{余子式}M'_{1j} \xrightarrow{\text{交换第}i-1\text{、}i\text{行}} D_n\text{的}(1,j)\text{余子式}M_{1j}$$

$$C = \overbrace{\sum_{j=1}^{n} a_{1j}(-1)^{1+j} \underbrace{M'_{1j}}_{-M_{1j}} = -\sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j}}^{\text{归纳假定}} = -D_n$$

交换行列式行/列，变号！（对调两行/列）

约定：r_i 表示 D_n 的第 i 行，$r_i \leftrightarrow r_k$ 表示交换 (i,k) 行

- 交换 D_n 的 (i,k) 行（$i<k$）：

$$\underbrace{\begin{array}{llll} r_i \leftrightarrow r_{i+1}, & r_{i+1} \leftrightarrow r_{i+2}, & \cdots & r_{k-1} \leftrightarrow r_k, \\ r_{k-1} \leftrightarrow r_{k-2}, & r_{k-2} \leftrightarrow r_{k-3}, & \cdots & , r_{i+1} \leftrightarrow r_i, \end{array} \begin{array}{l} k-i\text{次} \\ k-i-1\text{次} \end{array}}_{2(k-i)-1\text{次行交换，变号！}}$$

- 交换 D_n 的 (i,k) 列（$i<k$），变号！
- $D_n(i,k)$ 行/列相同，其值为 0！

$$\text{交换}(i,k)\text{行/列，}D_n\text{未变化，}D_n=(-1)D_n \quad \Rightarrow \quad D_n=0$$

- 行列式按任意行/列展开

$$D_n=\sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij}=\sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij}$$

行列式按任意行/列展开

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{aligned} &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \\ &= \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \end{aligned}$$

$$\begin{cases} r_i \leftrightarrow r_{i-1} \\ r_{i-1} \leftrightarrow r_{i-2} \\ \vdots \\ r_2 \leftrightarrow r_1 \\ i-1\text{次行交换} \end{cases}$$

将第 i 行移到第 1 行，不改变其余行的相对位置，

$$(-1)^{i-1}D_n = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \overbrace{\underbrace{\sum_{j=1}^{n} a_{ij}(-1)^{1+j}M_{ij}}}^{=(-1)^{i-1}D_n}$$

$$\Rightarrow D_n = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij}$$

行列式按任意行/列展开（续）

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{aligned} &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \\ &= \sum_{i=1}^{n} a_{ij}(-1)^{i+j}M_{ij} \end{aligned} \quad \begin{cases} r_i \leftrightarrow r_{i-1} \\ r_{i-1} \leftrightarrow r_{i-2} \\ \vdots \\ r_2 \leftrightarrow r_1 \\ i-1\text{次行交换} \end{cases}$$

将第 i 行移到第 1 行，不改变其余行的相对位置，

$$(-1)^{i-1}D_n = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \overbrace{\underbrace{\sum_{j=1}^{n} a_{ij}(-1)^{1+j}M_{ij}}_{\Rightarrow D_n=\sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij}}}^{=(-1)^{i-1}D_n}$$

行列式按任意行/列展开（续）

$$\overbrace{\sum_{j=1}^{n} a_{kj}A_{ij} = 0,}^{k\neq i} \qquad \overbrace{\sum_{i=1}^{n} a_{im}A_{ij} = 0}^{m\neq j} \qquad A_{ij} = (-1)^{i+j}M_{ij}$$

$$\begin{array}{llll} a_{i1} := a_{k1}, & a_{i2} := a_{k2}, & \cdots, & a_{in} := a_{kn} \\ A_{i1}, & A_{i2}, & \cdots, & A_{in} \end{array}$$

← 替换
← 不变

$$\sum_{j=1}^{n} a_{kj}A_{ij} = 0 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

← i行
按第i行展开
← k行
两行相同，行列式为 0

提取行列式一行/列的公因子

$$\underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{\sum_{j=1}^{n}(ca_{ij})(-1)^{i+j}M_{ij}} = \underbrace{c\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{c\sum_{j=1}^{n}a_{ij}(-1)^{i+j}M_{ij}}$$

$$0 = \begin{vmatrix} \vdots & \vdots & \cdots & \vdots \\ ca_{k1} & ca_{k2} & \cdots & ca_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix}$$

$\leftarrow$ i行，提取公因子

$\leftarrow$ k行 两行相同，行列式为 0

行列式两行/列成比例，其值为 0

行列式的拆分

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n}(a_{ij}+b_{ij})A_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij} + \sum_{j=1}^{n} b_{ij}A_{ij}$$

$$= \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{\sum_{j=1}^{n} a_{ij}A_{ij}} + \underbrace{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{\sum_{j=1}^{n} b_{ij}A_{ij}}$$

$|a_{ij}+b_{ij}|_{n\times n}$ 若每个元素都是两个数的和，则拆分为 2^n 个行列式的和！

行列式一行/列的 c 倍加到另一行/列，其值不变

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1}+ca_{i1} & a_{k2}+ca_{i2} & \cdots & a_{kn}+ca_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$r_k := r_k + cr_i$

$\leftarrow$ i行，r_i

$\leftarrow$ k行变动，r_k
其余行不变

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ ca_{i1} & ca_{i2} & \cdots & ca_{in} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix}$$

$= 0$
两行元素
成比例

计算行列式的一般方法

将行列式划归为“三角型”：

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \cdots = (-1)^s c \underbrace{\begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{vmatrix}}_{=b_{11}b_{22}\cdots b_{nn}} \quad \checkmark$$

- 交换行列式的两行/列（其值变号）；
- 提取行列式一行/列的公因子；
- 行列式一行/列的 c 倍加到另一行/列（其值不变）。

线性方程组与行列式

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \Bigg\| \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\Delta_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$j = 1, 2, \cdots, n$

$\Delta \neq 0$

$x_j = \dfrac{\Delta_j}{\Delta}$

$$a_{i1}\frac{\Delta_1}{\Delta} + a_{i2}\frac{\Delta_2}{\Delta} + \cdots + a_{in}\frac{\Delta_n}{\Delta} = b_i??? \qquad (i = 1, 2, \cdots, n)$$

克莱姆法则求解线性方程组

$$\sum_{j=1}^{n} a_{ij}\frac{\Delta_j}{\Delta} = \frac{1}{\Delta}\sum_{j=1}^{n} a_{ij}\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$\vdots = \frac{1}{\Delta}\sum_{j=1}^{n} a_{ij}\sum_{k=1}^{n} b_k A_{kj}$$ Δ_j按第j列展开（余子式不变）

$$\vdots = \frac{1}{\Delta}\sum_{k=1}^{n} b_k\sum_{j=1}^{n} a_{ij}A_{kj}$$ 交换求和顺序，$\sum_{j=1}^{n} a_{ij}A_{ij} = \Delta$

$$b_i = \frac{1}{\Delta}b_i\sum_{j=1}^{n} a_{ij}A_{ij} = \frac{1}{\Delta}b_i\Delta$$ 当$k \neq i$时，$\sum_{j=1}^{n} a_{ij}A_{kj} = 0$

克莱姆法则给出了唯一解

$$
\begin{array}{llll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & \times A_{1j} & \displaystyle\sum_{i=1}^{n} a_{ik}A_{ij} = 0 (k \neq j) \\
\cdots\ \cdots\ \cdots\ \cdots\cdots\ \cdots\ \cdots\ \cdots & \vdots & \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_2 & \times A_{ij} & \displaystyle\sum_{i=1}^{n} a_{ij}A_{ij} = \Delta \\
\cdots\ \cdots\ \cdots\ \cdots\cdots\ \cdots\ \cdots\ \cdots & \vdots & \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n & \times A_{nj} & \displaystyle\sum_{i=1}^{n} b_i A_{ij} = \Delta_i
\end{array}
$$

$$
\begin{array}{ccccccccc}
a_{11}A_{1j}x_1 & + & a_{12}A_{1j}x_2 & + & \cdots & + & a_{1n}A_{1j}x_n & = & b_1A_{1j} \\
a_{21}A_{2j}x_1 & + & a_{22}A_{2j}x_2 & + & \cdots & + & a_{2n}A_{2j}x_n & = & b_2A_{2j} \\
+ & + & + & + & \cdots & + & + & = & + \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & = & \vdots \\
+ & + & + & + & \cdots & + & + & = & + \\
a_{n1}A_{nj}x_1 & + & a_{n2}A_{nj}x_2 & + & \cdots & + & a_{nn}A_{nj}x_n & = & b_nA_{nj}
\end{array}
$$

克莱姆法则给出了唯一解（续）

$$\begin{array}{ccccccccccc} a_{11}A_{1j}x_1 & + & a_{12}A_{1j}x_2 & + & \cdots & + & a_{1n}A_{1j}x_n & = & b_1A_{1j} \\ a_{21}A_{2j}x_1 & + & a_{22}A_{2j}x_2 & + & \cdots & + & a_{2n}A_{2j}x_n & = & b_2A_{2j} \\ + & + & + & + & \cdots & + & + & = & + \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & = & \vdots \\ + & + & + & + & \cdots & + & + & = & + \\ a_{n1}A_{nj}x_1 & + & a_{n2}A_{nj}x_2 & + & \cdots & + & a_{nn}A_{nj}x_n & = & b_nA_{nj} \end{array}$$

$$0+\cdots+o+\Delta x_j+0+\cdots+0=\Delta_j \quad \Rightarrow \quad x_j=\frac{\Delta_j}{\Delta}$$

- $\Delta \neq 0$，$x_j=\Delta_j/\Delta \ (1\leq j\leq n)$ 必定是方程组的解，解存在；
- $\Delta \neq 0$，方程组的解必为 $x_j=\Delta_j/\Delta \ (1\leq j\leq n)$，解唯一。