# 第二章　矩阵与向量

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# 矩阵与向量的概念

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

矩阵/表格　向量/有序数组

$m = 1$　矩阵 = 行向量

$n = 1$　矩阵 = 列向量

$$\underbrace{vec(A) = (\overbrace{a_{11}, a_{21}, \cdots, a_{m1}}^{A\text{的第}1\text{列}}, \overbrace{a_{12}, a_{22}, \cdots, a_{m2}}^{A\text{的第}2\text{列}}, \cdots\cdots, \overbrace{a_{1n}, a_{2n}, \cdots, a_{mn}}^{A\text{的第}n\text{列}})_T}_{\text{矩阵的向量化}}$$

- 零矩阵（0 的地位?）与单位矩阵（1 的地位?）

$$0_{m\times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}_{m\times n}, \quad I_n = E_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

# 矩阵与向量的概念（续）

- 零向量、单位坐标向量、幺向量

$$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \epsilon_i = e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \leftarrow \ i-th \\ \\ (1 \le i \le n) \end{matrix}$$

- 对角矩阵 $diag(c_1, c_2, \cdots, c_n)$、数量矩阵

$$\begin{pmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{pmatrix}, \qquad \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$$

# 矩阵与向量的概念（续）

- 上三角矩阵、下三角矩阵

“扁”
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} & a_{1,m+1} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2m} & a_{2,m+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{mm} & a_{m,m+1} & \cdots & a_{mn} \end{pmatrix}$$

“瘦”
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \ddots & a_{nn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

上三角矩阵：
$i > j \;\Rightarrow\; a_{ij} = 0$

下三角矩阵：
$i < j \;\Rightarrow\; a_{ij} = 0$

# 矩阵的运算（一元运算，行列式与转置）

同型矩阵才可能相等：$(a_{ij})_{m\times n} = (b_{ij})_{m\times n}$，若 $a_{ij} = b_{ij}\ (\forall i, j)$

- 矩阵的转置，$(A^T)^T = A$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

对称：$A = A^T$
反对称：$A = -A^T$

$$\begin{matrix} A' \\ A^T \end{matrix} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}_{n\times m}$$

$(A^T)_{ij} = a_{ji}$
对称、反对称
均为方阵！

- 方阵的行列式：$|(a_{ij})_{n\times n}| = |A| = det(A)$；$|A^T| = |A|$

# 矩阵的运算（续，加法）

- 矩阵的加法，两个同型矩阵才能相加：

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

$A+B=B$

$A\pm 0=A$

$(-a_{ij})_{m\times n}$

↑ 负矩阵

$(A+B)+C=A+(B+C)$ $\quad (A+B)^T=A^T+B^T$

$|A+B|\neq |A|+|B|$! $\quad -A=0-A$ 负矩阵

$A-B:=A+(-B)$ $\quad$ 减法 $=$ 加法与数乘复合

# 矩阵的运算（续，数乘）

- 矩阵的数乘（一个数 $\lambda$ 与一个矩阵 $A$ 的运算）:

$$\lambda\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}$$

$$\lambda(A+B) = \lambda A + \lambda B \qquad 0\cdot A = 0_{m\times n},\quad 1\cdot A = A$$

$$(\lambda+\mu)A = \lambda A + \mu A \qquad (-1)\cdot A = -A \quad \leftarrow \text{负矩阵}$$

$$\lambda(\mu A) = (\lambda\mu)A \qquad (\lambda A)^T = \lambda A^T$$

$$\begin{vmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nn} \end{vmatrix} = \lambda^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

# 矩阵的运算（续，乘法）

- 矩阵的乘法，左矩阵的列数 = 右矩阵行数：

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{pmatrix}$$

$a_{i1},\ a_{i2},\ \cdots,\ a_{in} \leftarrow$ $A$的$i$行
$b_{1j},\ b_{2j},\ \cdots,\ b_{nj} \leftarrow$ $B$的$j$列

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}$$

$AB \neq BA$，举反例？ $\lambda(AB) = (\lambda A)B = A(\lambda B)$
$(A+C)B = AB + CB$ $A(B+C) = AB + AC$
$A0_{n\times k} = 0_{m\times k}$，$0_{k\times m}A = 0_{k\times n}$ $AE_n = A$，$E_m A = A$

# 矩阵的运算（续，乘法结合律、乘法与转置）

- $(a_{ij})_{m\times n}$，$(b_{ij})_{n\times k}$，$(c_{ij})_{k\times r}$，$(AB)C = A(BC)$

$$
\begin{aligned}
[(AB)C]_{ij} &= \sum_{s=1}^{k}(AB)_{is}c_{sj} = \sum_{s=1}^{k}\overbrace{\sum_{t=1}^{n}a_{it}b_{ts}}^{(AB)_{is}}c_{sj} \\
&= \sum_{t=1}^{n}a_{it}\underbrace{\sum_{s=1}^{k}b_{ts}c_{sj}}_{(BC)_{tj}} = [A(BC)]_{ij}
\end{aligned}
$$

- $A = (a_{ij})_{m\times n}$，$B = (b_{ij})_{n\times k}$，$(AB)^T = B^T A^T$

$$
\begin{aligned}
[(AB)^T]_{ij} &= (AB)_{ji} = \sum_{s=1}^{n}a_{js}b_{si} \qquad a_{js} = (A^T)_{sj}, \\
&= \sum_{s=1}^{n}(B^T)_{is}(A^T)_{sj} = (B^T A^T)_{ij} \quad b_{si} = (B^T)_{is}
\end{aligned}
$$

# 矩阵的运算（续，乘法与行列式-1）

- $A=(a_{ij})_{n\times n}$，$B=(b_{ij})_{n\times n}$，$|AB|=|A||B|$，记 $C=AB$

$$|A||B|=\left|\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 \\ -- & -- & -- & -- & -- & -- & -- & -- \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \\ a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{array}$$

$$\begin{array}{l} r_1+a_{11}r_{n+1} \\ r_1+a_{12}r_{n+2} \\ \vdots \\ r_1+a_{1n}r_{n+n} \end{array} \Rightarrow 0\ 0\ \cdots\ 0\ \overbrace{\sum_{j=1}^{n}a_{1j}b_{j1}}^{c_{11}=(AB)_{11}}\ \overbrace{\sum_{j=1}^{n}a_{1j}b_{j2}}^{c_{12}=(AB)_{12}}\ \cdots\ \overbrace{\sum_{j=1}^{n}a_{1j}b_{jn}}^{c_{1n}=(AB)_{1n}}$$

# 矩阵的运算（续，乘法与行列式-2）

$$|A||B| = \left|\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 \\ -- & -- & -- & -- & -- & -- & -- & -- \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \\ a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{array}$$

$$\begin{array}{l} r_2 + a_{21} r_{n+1} \\ r_2 + a_{22} r_{n+2} \\ \vdots \\ r_2 + a_{2n} r_{n+n} \end{array} \Rightarrow 0\ 0\ \cdots\ 0\ \overbrace{\sum_{j=1}^{n} a_{2j} b_{j1}}^{c_{21}=(AB)_{21}}\ \overbrace{\sum_{j=1}^{n} a_{2j} b_{j2}}^{c_{22}=(AB)_{22}}\ \cdots\ \overbrace{\sum_{j=1}^{n} a_{2j} b_{jn}}^{c_{2n}=(AB)_{2n}}$$

# 矩阵的运算（续，乘法与行列式-3）

$$|A||B| = \left|\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 0 \\ -- & -- & -- & -- & -- & -- & -- & -- \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right| \begin{array}{c} \\ \\ \\ \\ \\ a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{array}$$

$$\begin{array}{l} r_n + a_{n1} r_{n+1} \\ r_n + a_{n2} r_{n+2} \\ \vdots \\ r_n + a_{nn} r_{n+n} \end{array} \Rightarrow 0 \ 0 \ \cdots \ 0 \ \overbrace{\sum_{j=1}^{n} a_{nj} b_{j1}}^{c_{n1}=(AB)_{n1}} \ \overbrace{\sum_{j=1}^{n} a_{nj} b_{j2}}^{c_{n2}=(AB)_{n2}} \ \cdots \ \overbrace{\sum_{j=1}^{n} a_{nj} b_{jn}}^{c_{nn}=(AB)_{nn}}$$

# 矩阵的运算（续，乘法与行列式-4）

$$|A||B| = \left|\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & 0 & \cdots & 0 & c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & c_{n1} & c_{n2} & \cdots & c_{nn} \\ -- & -- & -- & -- & -- & -- & -- & -- \\ -1 & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & -1 & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right|$$

$$\begin{array}{ccc} c_1 & \leftrightarrow & c_{n+1} \\ c_2 & \leftrightarrow & c_{n+2} \\ \vdots & & \\ c_n & \leftrightarrow & c_{n+n} \end{array} \Rightarrow \quad (-1)^n|A||B| = \begin{vmatrix} C & 0 \\ B & -E_n \end{vmatrix} = (-1)^n|C|$$

$$|A||B| = |C| = |AB|$$

# 矩阵的运算（续，向量相乘）

- 行向量乘以列向量 = 数（内积？）

$$\underbrace{(y_1, y_2, \cdots, y_n)}_{=y^T}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = y^Tx$$

- 列向量乘以行向量 = 矩阵（秩一？）

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}(y_1, y_2, \cdots, y_n) = \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \cdots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{pmatrix} = xy^T$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}\overbrace{y_1}^{1\times 1} = y_1\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

矩阵乘法 = 数乘

约定：向量 = 列向量

# 矩阵的运算（续，矩阵向量相乘）

- 线性方程组的矩阵表示 $Ax = b$

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- $Ae_j =$ 第 $j$ 列，$e_i^T A =$ 第 $i$ 行

$$Ae_j = (a_{1j}, a_{2j}, \cdots, a_{mj})^T, \quad e_i^T A = (a_{i1}, a_{i2}, \cdots, a_{in})$$

# 矩阵的运算（续，方阵的乘幂、矩阵函数）

- $A=(a_{ij})_{n\times n}$，$f(t)=a_0+a_1t+\cdots+a_kt^k$，矩阵多项式？

$$\begin{aligned} A^m &= \overbrace{A\times A\times\cdots\times A}^{m\text{ 个 }A},\quad A^0=E_n \\ f(A) &= a_0E_n+a_1A+\cdots+a_kA^k \end{aligned}$$

$f(t)=a_0+a_1t+\cdots+a_kt^k$，$g(t)=b_0+b_1t+\cdots+b_mt^m$，
$h(t)=f(t)g(t)$（$m+k$ 次多项式）

$$f(A)^T=f(A^T),\quad h(A)=f(A)g(A)=g(A)f(A)$$

- 多项式的商称为有理函数，有理矩阵函数？一般函数 $\sin(t),\cdots$？可交换 $f(A)g(A)=g(A)f(A)$？

$$f(t)=\frac{a_0+a_1t+\cdots+a_kt^k}{b_0+b_1t+\cdots+b_mt^m},\quad h(A)?$$

# 矩阵的逆与除法（逆矩阵的引入）

除法 $=$ 乘法与倒数复合。（1）推广倒数；（2）定义除法，左右？

数 $a, b$ $\left|\begin{array}{l} a\times b=1 \\ b\times a=1 \end{array}\right.$ $\Rightarrow\ b=\dfrac{1}{a}=a^{-1},\ a=\dfrac{1}{b}=b^{-1}$ 倒数/逆

$$\begin{array}{c} A_{m\times n} \\ B_{n\times k} \end{array}\left|\begin{array}{l} \overbrace{AB}^{m\times k}=E_m \\ \quad\Downarrow \\ \underbrace{BA}_{n\times n}=E_n \end{array}\right. \Rightarrow \begin{array}{l} m=k \\ m>n,\ AB\neq E_m \\ m>n,\ BA\neq E_n \end{array} \Rightarrow m=n=k$$

给定 $A=(a_{ij})_{n\times n}$ 与 $B=(b_{ij})_{n\times n}$（方阵），

$$AB=BA=E_n \quad\Leftrightarrow\quad B=A^{-1},\quad A=B^{-1}$$ 逆矩阵

若 $A$ 有逆矩阵，则称之为可逆，核心概念，存在、唯一？计算？

# 逆矩阵的存在唯一性

给定 $A=(a_{ij})_{n\times n}$（方阵），若可逆，即存在 $B=A^{-1}$

$$\begin{matrix} AB \\ BA \end{matrix} = E \quad \Rightarrow \quad |A||B| = |AB| = |E| = 1 \quad \Rightarrow \quad |A^{-1}| = \frac{1}{|A|}$$

- 如果 $A$ 可逆，则 $|A|\neq 0$；（是否充分呢？）

伴随矩阵

$$A^* = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n1} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}A^*,\quad (A^*)^{-1} = \frac{1}{|A|}A$$

$$|AA^*| = ||A|E_n| = |A|^n$$

$$|A^*| = |A|^{n-1}$$

$$\left.\begin{aligned} (AA^*)_{ij} &= \sum_{k=1}^{n} a_{ik}\overbrace{(A^*)_{kj}}^{A_{jk}} \\ (A^*A)_{ij} &= \sum_{k=1}^{n} \underbrace{(A^*)_{ik}}_{A_{ki}} a_{kj} \end{aligned}\right\} = \begin{cases} |A| & i=j \\ 0 & i\neq j \end{cases}$$

$$\overbrace{AA^* = A^*A}^{=|A|E_n}$$

$$\underbrace{A\frac{1}{|A|}A^* = \frac{1}{|A|}A^*A}_{=E_n}$$

# 逆矩阵的存在唯一性（续）

- $|A|=0$，$A$ 没有逆矩阵，称为不可逆、奇异、退化；
- $|A|\neq 0$，$A$ 有逆矩阵；$A_1, A_2, \cdots, A_k$ 均为方阵，$A=A_1A_2\cdots A_k$ 逆矩阵当且仅当每个 $A_i$（$1\le i\le k$）可逆，

$$A_k^{-1}\cdots \overbrace{A_2^{-1}\underbrace{A_1^{-1}A_1}_{E_n}A_2}^{=E_n}\cdots A_k = E_n$$

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$$

---

$$\begin{aligned}
A_{n\times n}B_{n\times n}=E_n \quad &\Rightarrow \quad |A|\neq 0,\ |B|\neq 0 \\
AB=E_n\text{或者} \quad &\Rightarrow \quad B=A^{-1}AB=A^{-1}E_n=A^{-1} \\
BA=E_n\text{，一个} \quad & \quad\ \ A=ABB^{-1}=E_nB^{-1}=B^{-1} \\
\text{条件即可保证}A\text{可逆} \quad &\Rightarrow \quad BA=E_n\ \text{先决条件}A,B\text{是方阵}
\end{aligned}$$

# 可逆矩阵与伴随矩阵（续）

- $A$ 可逆，$\lambda \neq 0$，则 $A^{-1}$、$A^*$ 与 $A^T$ 可逆，

$$
\begin{aligned}
A^T(A^{-1})^T = (A^{-1}A)^T = E_n^T = E_n &\implies (A^T)^{-1} = (A^{-1})^T \\
\lambda A \lambda^{-1} A^{-1} = AA^{-1} = E_n &\implies (\lambda A)^{-1} = \lambda^{-1}A^{-1} \\
AA^{-1}(A^{-1})^T = E_n(A^{-1})^T &\implies (A^T)^{-1} = (A^{-1})^T \\
A\text{不可逆 } A^*\text{，}(A^*)^* \text{？} \quad A^{-1} = \frac{1}{|A|}A^* &\implies (A^*)^{-1} = \frac{1}{|A|}A
\end{aligned}
$$

- 上（下）三角矩阵的逆矩阵是上（下）三角的；
- 对称矩阵的逆矩阵是对称的；

$$
A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}
$$

观察，有对角元为 0

$$
\overbrace{i < j \ \Rightarrow \ M_{ij} = 0}^{\text{观察，有对角元为 0}}
$$

$$
\Rightarrow \underbrace{A_{ij} = (-1)^{i+j}M_{ij} = 0}_{A^*\text{下三角部分}}
$$

$$
\Rightarrow A^*\text{上三角}
$$

# 矩阵的分块

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \left(\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1k} \\ \hline A_{21} & A_{22} & \cdots & A_{2k} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline A_{s1} & A_{s2} & \cdots & A_{sk} \end{array}\right) \begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{matrix}$$

$$n_1 \quad n_2 \quad \cdots \quad n_k$$

$$m_1 + m_2 + \cdots + m_s = m$$

$$n_1 + n_2 + \cdots + n_k = n$$

$A_{pq} = (a_{ij})_{m_p \times n_q}$ 不是代数余子式

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ -- & -- & -- & -- \\ a_{21} & a_{22} & \cdots & a_{2n} \\ -- & -- & -- & -- \\ \vdots & \vdots & \cdots & \vdots \\ -- & -- & -- & -- \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{matrix} \leftarrow \beta_1^T \\ \\ \leftarrow \beta_2^T \\ \\ \vdots \\ \\ \leftarrow \beta_m^T \end{matrix}$$

$$\beta_i^T = (a_{i1}, a_{i2}, \cdots, a_{in})$$

按 $m_1 = \cdots = m_s = 1$

行 $n_1 = n$

分 $s = m,\ ,\ k = 1$

块 $m$个$n$维行向量

# 矩阵的分块（续）

$$\begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_k \end{array}$$

$$\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right)$$

按列分块

$n_1 = \cdots = n_k = 1$

$m_1 = m$

$s = 1, \ , k = n$

$n$个$m$维列向量

$$\alpha_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

- 分块矩阵的转置运算

$$\left(\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1k} \\ \hline A_{21} & A_{22} & \cdots & A_{2k} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline A_{s1} & A_{s2} & \cdots & A_{sk} \end{array}\right)^T = \begin{array}{c} \begin{pmatrix} A_{11}^T & A_{21}^T & \cdots & A_{s1}^T \\ A_{12}^T & A_{22}^T & \cdots & A_{22}^T \\ \vdots & \vdots & \cdots & \vdots \\ A_{1k}^T & A_{2k}^T & \cdots & A_{sk}^T \end{pmatrix} \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_k \end{array} \\ \begin{array}{cccc} m_1 & m_2 & \cdots & m_s \end{array} \end{array}$$

# 矩阵的分块（续，加法）

- 分块矩阵的加法与数乘运算

$$\begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{matrix}\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sk} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{sk} \end{pmatrix}\begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{matrix}$$

$$\begin{matrix} n_1 & n_2 & \cdots & n_k \end{matrix} \qquad \begin{matrix} n_1 & n_2 & \cdots & n_k \end{matrix}$$

$$= \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1k}+B_{1k} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2k}+B_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ A_{s1}+B_{s1} & A_{s2}+B_{s2} & \cdots & A_{sk}+B_{sk} \end{pmatrix}$$

同型矩阵

分块方式相同

$$\lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sk} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1k} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda A_{s1} & \lambda A_{s2} & \cdots & \lambda A_{sk} \end{pmatrix}$$

# 矩阵的分块（续，乘法）

- 分块矩阵的乘法运算

$$\begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{matrix} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{sk} \end{pmatrix} \times \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{sk} \end{pmatrix} \begin{matrix} n_1 \\ n_2 \\ \vdots \\ n_k \end{matrix}$$

$$\qquad n_1 \quad n_2 \quad \cdots \quad n_k \qquad\qquad r_1 \quad r_2 \quad \cdots \quad r_p$$

$$= \begin{matrix} m_1 \\ m_2 \\ \vdots \\ m_s \end{matrix} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1p} \\ C_{21} & C_{22} & \cdots & C_{2p} \\ \vdots & \vdots & \cdots & \vdots \\ C_{s1} & C_{s2} & \cdots & C_{sp} \end{pmatrix}$$

$$\qquad r_1 \quad r_2 \quad \cdots \quad r_p$$

$A$的列数 $=$ $B$的行数
$A$的列的划分 $=$ $B$的行的划分
$C$的行的划分 $=$ $A$的行的划分
$C$的列的划分 $=$ $B$的列的划分

检查两个矩阵是否可以相乘，两个矩阵分块方式是否匹配

# 矩阵的分块（续，常用的分块相乘）

$$AB = \overbrace{\begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_m^T \end{pmatrix}}^{A\text{按行分块}} \times B = \begin{pmatrix} \alpha_1^T B \\ \alpha_2^T B \\ \vdots \\ \alpha_m^T B \end{pmatrix}$$

$$\alpha_i^T \beta_j = \sum_{s=1}^{n} a_{is} b_{sj}$$

$$\underbrace{\left( \sum_{s=1}^{n} a_{is} b_{s1}, \cdots, \sum_{s=1}^{n} a_{is} b_{sk} \right)}_{=\alpha_i^T B \quad AB\text{的一行}}$$

$$\begin{aligned} AB &= A \overbrace{(\beta_1, \beta_2, \cdots, \beta_k)}^{\text{按列分块}} \\ &= (A\beta_1, A\beta_2, \cdots, A\beta_k) \end{aligned}$$

$$A\beta_j = \begin{pmatrix} \sum_{s=1}^{n} a_{1s} b_{sj} \\ \vdots \\ \sum_{s=1}^{n} a_{ms} b_{sj} \end{pmatrix} \quad AB \text{ 的一列}$$

($e_i^T B$：取出$B$的第$i$行) $E_m B = \begin{pmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{pmatrix}$,

($Ae_j$：取出$A$的第$j$列)

$AE_n = (\alpha_1, \alpha_2, \cdots, \alpha_n)$,

# 矩阵的分块（续，常用的分块相乘）

$$AB=\overbrace{\begin{pmatrix}\alpha_1^T\\ \alpha_2^T\\ \vdots\\ \alpha_m^T\end{pmatrix}}^{\text{按行分块}}\times\underbrace{(\beta_1,\beta_2,\cdots,\beta_k)}_{B\ \text{按列分块}}=\begin{pmatrix}\alpha_1^T\beta_1 & \alpha_1^T\beta_2 & \cdots & \alpha_1^T\beta_k\\ \alpha_2^T\beta_1 & \alpha_2^T\beta_2 & \cdots & \alpha_2^T\beta_k\\ \vdots & \vdots & \cdots & \vdots\\ \alpha_m^T\beta_1 & \alpha_m^T\beta_2 & \cdots & \alpha_m^T\beta_k\end{pmatrix}$$

$\alpha_i^T\beta_j=\text{数}_{1\times1}$

$$AB=\overbrace{\underbrace{(\alpha_1,\alpha_2,\cdots,\alpha_n)}_{\text{按列分块}}\times\underbrace{\begin{pmatrix}\beta_1^T\\ \beta_2^T\\ \vdots\\ \beta_n^T\end{pmatrix}}_{A\text{按行分块}}}^{=\alpha_1\beta_1^T+\alpha_2\beta_2^T+\cdots+\alpha_n\beta_n^T,\ n\text{个}m\times k\text{矩阵之和}}$$

$$\overbrace{\begin{pmatrix}a_{1i}b_{i1} & a_{1i}b_{i2} & \cdots & a_{1i}b_{ik}\\ a_{2i}b_{i1} & a_{2i}b_{i2} & \cdots & a_{2i}b_{ik}\\ \vdots & \vdots & \cdots & \vdots\\ a_{mi}b_{i1} & a_{mi}b_{i2} & \cdots & a_{mi}b_{ik}\end{pmatrix}}^{=\alpha_i\beta_i^T,\ \text{秩 1 矩阵？}}$$

# 分块矩阵相乘（续，向量组的线性组合与线性方程组）

$$A=\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right) \begin{array}{c} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_m^T \end{array} \qquad x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad y=\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\alpha_1 \qquad \alpha_2 \qquad \cdots \qquad \alpha_n$$

$Ax = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ ↑ $x_j, y_i$称为乘子

$y^T A = y_1\beta_1^T + y_2\beta_2^T + \cdots + y_m\beta_m^T$ 向量组的线性组合

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ \hline x_1\cdot\alpha_1 \qquad x_2\cdot\alpha_2 \qquad\qquad x_n\cdot\alpha_n \qquad b \end{array}\right.$$

$x = (x_1, x_2, \cdots, x_n)^T$

$A = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_n)$

$\underbrace{x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n}_{=Ax}$

$Ax = 0/b \neq 0$ 齐次/非齐次

# 矩阵的分块（续，排列矩阵）

$$e_i = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad P_n = \begin{pmatrix} e_{i_1}^T \\ e_{i_2}^T \\ \vdots \\ e_{i_n}^T \end{pmatrix} = (e_{i_2}, e_{i_2}, \cdots, e_{i_n}) \leftarrow \text{排列矩阵}$$

$|P_n| = \pm 1$ 奇偶排列，排列的逆序数

重排$1, 2, \cdots, n \rightarrow i_1, i_2, \cdots, i_n$

$$P_m A = \begin{pmatrix} e_{i_1}^T \\ e_{i_2}^T \\ \vdots \\ e_{i_m}^T \end{pmatrix} A = \begin{pmatrix} e_{i_1}^T A \\ e_{i_2}^T A \\ \vdots \\ e_{i_m}^T A \end{pmatrix} = \begin{pmatrix} a_{i_1}^T \\ a_{i_2}^T \\ \vdots \\ a_{i_m}^T \end{pmatrix}$$

$e_i^T A = a_i^T$：$A$的第$i$行

$\leftarrow$ $A$的行向量重排

$$AP_n = A(e_{i_1}, e_{i_2}, \cdots, e_{i_n}) = \overbrace{(Ae_{i_1}, Ae_{i_2}, \cdots, Ae_{i_n})}^{Ae_j = a_i^T : A\text{的第}j\text{列}} = \overbrace{(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n})}^{A\text{的列向量重排}}$$

# 初等变换与初等矩阵

- 第 $i$ 行/列与第 $k$ 行/列互换（位置对调）：$r_i \leftrightarrow r_k / c_i \leftrightarrow c_k$
- 常数 $\lambda \neq 0$ 乘以第 $i$ 行/列（每一个元素）：$\lambda \times r_i / \lambda \times c_i$
- 第 $j$ 行/列的 $\lambda$ 倍加到第 $i$ 行/列：$r_i + \lambda r_j / c_i + \lambda c_j$

$$\overbrace{(e_1 \cdots e_i \cdots e_k \cdots e_n)}^{=E_n\text{，单位矩阵}} = \begin{pmatrix} e_1^T \\ \vdots \\ e_i^T \\ \vdots \\ e_k^T \\ \vdots \\ e_n^T \end{pmatrix} \xrightarrow{r_i \leftrightarrow r_k} \text{或者} \xrightarrow{c_i \leftrightarrow c_k} \underbrace{\left(\begin{array}{ccccccc|c} 1 & & & & & & & e_1^T \\ & \ddots & & & & & & \vdots \\ & & 0 & \cdots & 1 & & & e_k^T \\ & & \vdots & \ddots & \vdots & & & \vdots \\ & & 1 & \cdots & 0 & & & e_i^T \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & e_n^T \\ \hline e_1 & \cdots & e_k & \cdots & e_i & \cdots & e_n & \end{array}\right)}_{=:E(i,k)\text{ 初等矩阵，对称，排列矩阵，}|E(i,k)|=-1}$$

# 初等变换与初等矩阵（续）

$$\underbrace{E(i,k)}_{m\times m} A = \begin{matrix} 1\text{行} \\ \vdots \\ i\text{行} \\ \vdots \\ k\text{行} \\ \vdots \\ n\text{行} \end{matrix} \begin{pmatrix} e_1^T \\ \vdots \\ e_k^T \\ \vdots \\ e_i^T \\ \vdots \\ e_n^T \end{pmatrix} A = \begin{pmatrix} e_1^T A \\ \vdots \\ e_k^T A \\ \vdots \\ e_i^T A \\ \vdots \\ e_n^T A \end{pmatrix} \begin{matrix} \\ \\ \leftarrow A\text{的第}k\text{个行向量} \\ \\ \leftarrow A\text{的第}i\text{个行向量} \\ \\ \\ \end{matrix}$$

交换了$A$的第$i$行与$k$行

$$\begin{aligned} A\underbrace{E(i,k)}_{n\times n} &= A\left(\begin{array}{ccccccc} e_1 & \cdots & e_k & \cdots & e_i & \cdots & e_n \\ \hline 1^{st} & \cdots & i^{th} & \cdots & k^{th} & \cdots & n^{th} \end{array}\right) \\ &= \left(\begin{array}{ccccccc} Ae_1 & \cdots & Ae_k & \cdots & Ae_i & \cdots & Ae_n \\ \hline 1^{st} & \cdots & i^{th} & \cdots & k^{th} & \cdots & n^{th} \end{array}\right) \\ &= \begin{pmatrix} \alpha_1 & \cdots & \alpha_k & \cdots & \alpha_i & \cdots & \alpha_n \end{pmatrix} \end{aligned}$$

$Ae_j = \alpha_j$ 是$A$的第$j$列

交换了$A$的第$i$列与$k$列

# 初等变换与初等矩阵（续）

$$\overbrace{(e_1 \cdots e_i \cdots e_n)}^{=E_n,\ \text{单位矩阵}} \Big\| \begin{pmatrix} e_1^T \\ \vdots \\ e_i^T \\ \vdots \\ e_n^T \end{pmatrix} \xrightarrow{\lambda r_i} \text{或者} \xrightarrow{\lambda c_i} \underbrace{\left(\begin{array}{ccccc|c} 1 & & & & & e_1^T \\ & \ddots & & & & \vdots \\ & & \lambda & & & \lambda e_i^T \\ & & & \ddots & & \vdots \\ & & & & 1 & e_n^T \\ \hline e_1 & \cdots & \lambda e_i & \cdots & e_n & \end{array}\right)}_{=:E(i(\lambda))\ \text{初等矩阵，对角，}|E(i(\lambda))|=\lambda}$$

$(\lambda \neq 0!!!)$

$$\underbrace{E(i(\lambda))}_{m\times m} A = \begin{pmatrix} e_1^T \\ \vdots \\ \lambda e_i^T \\ \vdots \\ e_n^T \end{pmatrix} A = \begin{pmatrix} e_1^T A \\ \vdots \\ \lambda e_i^T A \\ \vdots \\ e_n^T A \end{pmatrix}$$

$E(i(\lambda))$左（右）乘$A$：$\lambda$乘以$A$的第$i$行（列）每个元素
$\leftarrow \lambda \cdot A$的第$k$个行向量

# 初等变换与初等矩阵（续）

$$\overbrace{(e_1 \cdots e_i \cdots e_n)}^{=E_n,\ \text{单位矩阵}} \ \| \ \begin{pmatrix} e_1^T \\ \vdots \\ e_i^T \\ \vdots \\ e_n^T \end{pmatrix} \xrightarrow{r_i+\lambda r_k} \text{或者} \xrightarrow{c_k+\lambda c_i} \underbrace{\left(\begin{array}{cccc|c} \ddots & & & & \\ & 1 & \cdots & \lambda & e_i^T+\lambda e_k^T \\ & & \ddots & \vdots & \\ & & & 1 & e_k^T \\ & & & & \ddots \\ \hline & e_i & \cdots & \alpha & \cdots & \alpha = e_k+\lambda e_i \end{array}\right)}_{=:E(i,k(\lambda))\ \text{初等矩阵，三角矩阵，}|E(i,k(\lambda))|=1}$$

$$E(i,k(\lambda))A = \begin{pmatrix} e_1^T A \\ \vdots \\ (e_i^T+\lambda e_k^T)A \\ \vdots \\ e_n^T A \end{pmatrix}$$

$E(i,k(\lambda))$左乘$A$： $A$第$k$行的$\lambda$倍加到$A$的第$i$行

$A(i.:)+\lambda A(k,:)$

# 初等变换与初等矩阵（续）

$$\overbrace{(e_1 \cdots e_i \cdots e_n)}^{=E_n,\ \text{单位矩阵}} = \begin{pmatrix} e_1^T \\ \vdots \\ e_i^T \\ \vdots \\ e_n^T \end{pmatrix} \xrightarrow{r_i+\lambda r_k} \text{或者} \xrightarrow{c_k+\lambda c_i} \underbrace{\left(\begin{array}{ccccc|c} \ddots & & & & & \\ & 1 & \cdots & \lambda & & e_i^T+\lambda e_k^T \\ & & \ddots & \vdots & & \\ & & & 1 & & e_k^T \\ & & & & \ddots & \\ \hline & e_i & \cdots & \alpha & \cdots & \alpha = e_k+\lambda e_i \end{array}\right)}_{=:E(i,k(\lambda))\ \text{初等矩阵，三角矩阵，}|E(i,k(\lambda))|=1}$$

$$E(i,k(\lambda))A = \begin{pmatrix} e_1^T A \\ \vdots \\ (e_i^T+\lambda e_k^T)A \\ \vdots \\ e_n^T A \end{pmatrix}$$

$E(i,k(\lambda))$左乘$A$： $A$第$k$行的$\lambda$倍加到$A$的第$i$行

$A(i.:)+\lambda A(k,:)$

$E(i,k(\lambda))$右乘$A$： $A$第$i$列的$\lambda$倍加到$A$的第$k$列

# 初等变换与初等矩阵（续）

$$E \xrightarrow[c_i \leftrightarrow c_k]{r_i \leftrightarrow r_k} E(i,k) \qquad A \xrightarrow{r_i \leftrightarrow r_k} E(i,k)A \qquad A \xrightarrow{c_i \leftrightarrow c_k} AE(i,k)$$

$$E \xrightarrow[\lambda c_i]{\lambda r_i} E(i(\lambda)) \qquad A \xrightarrow{\lambda r_i} E(i(\lambda))A \qquad A \xrightarrow{\lambda c_i} AE(i(\lambda))$$

$$E \xrightarrow[c_k+\lambda c_i]{r_i+\lambda r_k} E(i,k(\lambda)) \qquad A \xrightarrow{r_i+\lambda r_k} E(i,k(\lambda))A \qquad A \xrightarrow{c_k+\lambda c_i} AE(i,k(\lambda))$$

---

$$E \xrightarrow{r_i \leftrightarrow r_k} E(i,k) \xrightarrow{r_i \leftrightarrow r_k} E \quad \Longrightarrow \quad E(i,k)E(i,k)E = E$$

$$E \xrightarrow{\lambda r_i} E(i(\lambda)) \xrightarrow{\lambda^{-1} r_i} E \quad \Longrightarrow \quad E(i(\lambda^{-1}))E(i(\lambda)) = E$$

$$E \xrightarrow{r_i+\lambda r_k} E(i,k(\lambda)) \xrightarrow{r_i-\lambda r_k} E \quad \Longrightarrow \quad E(i,k(\lambda))E(i,k(-\lambda)) = E$$

$$E(i,k)^{-1} = E(i,k) \qquad E(i(\lambda))^{-1} = E(i(\lambda^{-1}))$$

$$E(i,k(\lambda))^{-1} = E(i,k(-\lambda))$$

初等矩阵可逆，一次行/列变换施加于$E$

# 矩阵的行、列等价性

- $A=(a_{ij})_{m\times n}$ 与 $B=(b_{ij})_{m\times n}$ “行等价”（$A\stackrel{r}{\rightarrow}B$），如果

$$A\underbrace{\xrightarrow{\text{行变换}}\cdots\xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)}B=P_k\cdots P_2P_1A$$

- $A=(a_{ij})_{m\times n}$ 与 $B=(b_{ij})_{m\times n}$ “列等价”（$A\stackrel{c}{\rightarrow}B$），如果

$$A\underbrace{\xrightarrow{\text{列变换}}\cdots\xrightarrow{\text{列变换}}}_{\text{对应初等矩阵}Q_j(1\le j\le s)}B=AQ_1Q_2\cdots Q_s$$

- $A=(a_{ij})_{m\times n}$ 与 $B=(b_{ij})_{m\times n}$ “等价”（$A\rightarrow B$），如果

$$A\underbrace{\xrightarrow{\text{行/列变换}}\cdots\xrightarrow{\text{行/列变换}}}_{\text{对应初等矩阵}P_i,Q_j}B=P_k\cdots P_2P_1AQ_1Q_2\cdots Q_s$$

# “行等价”、“列等价”与“等价”为“等价关系”

- **自反性：** $A \overset{c}{\to} A$

$$A = P_1^{-1} P_1 A \quad P_1,\ P_1^{-1}\text{初等矩阵}$$

- **对称性：** 如果 $A \overset{c}{\to} B$，则 $B \overset{c}{\to} A$

$$B = P_1 P_2 \cdots P_k A \ \Rightarrow\ A = (P_1 P_2 \cdots P_k)^{-1} B = \underbrace{P_k^{-1} \cdots P_2^{-1} P_1^{-1}}_{P_i^{-1}\ \text{初等矩阵}} B$$

- **传递性：** 如果 $A \overset{c}{\to} B$，$B \overset{c}{\to} C$，则 $A \overset{c}{\to} C$

$$A \overset{c}{\to} B \ \Rightarrow\ \overbrace{B = P_k \cdots P_2 P_1 A}^{P_i\ \text{初等矩阵}},\quad \overbrace{C = Q_s \cdots Q_2 Q_1 B}^{Q_j\ \text{初等矩阵}}$$

$$\Rightarrow \quad C = Q_s \cdots Q_2 Q_1 P_k \cdots P_2 P_1 A$$

# 行简化梯形矩阵

有限次行变换，能将矩阵 $A=(a_{ij})_{m\times n}$ 简化到什么程度？

$$A \underbrace{\xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\leq i\leq k)} B=P_k\cdots P_2P_1A \text{ 行简化梯形矩阵}$$

- $B$ 的零行均在非零行的下方，

$B(i,:)$ 为零行，如果： $b_{i1}=b_{i2}=\cdots=b_{in}=0$

- $B$ 的非零行自左向右第一个非零元，其位置为，

$(1,j_1)$， $(2,j_2)$， $\cdots$， $(r,j_r)$， $1\leq j_1<j_2<\cdots<j_r\leq n$

- $B$ 的非零行自左向右第一个非零元为 1（“主 1”），
- 主 1 所在列中其余元素（$m-1$ 个）均为 0。

# 行简化梯形矩阵（续）

$$A \underbrace{\xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)} B = P_k \cdots P_2 P_1 A \quad \text{行简化梯形矩阵}$$

$B$ 有 $r$ 个非零行，即 $B$ 有 $r$ 个 “主 1”，$(1, j_1), (2, j_2), \cdots, (r, j_r)$

$$1 \le j_1 < j_2 < \cdots < j_r \le n \quad j_1 = 1, j_2 = 2, \cdots, j_r = r$$

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,r+1} & b_{1,r+2} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{2,r+1} & b_{2,r+2} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{r,r+1} & b_{r,r+2} & \cdots & b_{rn} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline e_1 & e_2 & \cdots & e_r & \beta_{r+1} & \beta_{r+2} & \cdots & \beta_n \end{pmatrix} \quad \beta_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

# 行简化梯形矩阵（续）

$$A \underbrace{\xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)} B = P_k \cdots P_2 P_1 A \quad \text{行简化梯形矩阵}$$

$B = (\beta_1 \ \beta_2 \ \cdots \ \beta_n)$ 有 $r$ 个 "主 1"：$(1, j_1), (2, j_2), \cdots, (r, j_r)$

$$1 \le j_1 < j_2 < \cdots < j_r \le n, \quad \beta_{j_1} = e_1, \ \beta_{j_2} = e_2, \ \cdots, \ \beta_{j_r} = e_r$$

$$\beta_j^T = (b_{1j}, \ b_{2j}, \ \cdots, \ b_{rj}, \ \overbrace{0, \ \cdots, \ 0}^{\text{位于零行}}) \quad = \quad b_{1j}e_1 + b_{2j}e_2 + \cdots + b_{rj}e_r$$
$$= \quad b_{1j}\beta_{j_1} + b_{2j}\beta_{j_2} + \cdots + b_{rj}\beta_{j_r}$$

- 有限次行变换必定可以将 $A$ 变换为 "行简化梯形矩阵"；换言之，$A$ 必定与某个行简化梯形矩阵行等价

$$A \xrightarrow{r} \text{行简化梯形矩阵}$$

# 列简化梯形矩阵

有限次列变换，能将矩阵 $A=(a_{ij})_{m\times n}$ 简化到什么程度？

$$A \underbrace{\xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}}}_{\text{对应初等矩阵}Q_j(1\leq j\leq s)} B = AQ_1Q_2\cdots Q_s$$ 列行简化梯形矩阵

- $B$ 的零列均在非零列的右方，

  $B(:,j)$ 为零列，如果： $b_{1j}=b_{2j}=\cdots=b_{mj}=0$

- $B$ 的非零列自上至下第一个非零元，其位置为，

  $(i_1,1),\quad (i_2,2),\quad \cdots,\quad (i_r,r),\quad 1\leq i_1<i_2<\cdots<i_r\leq m$

- $B$ 的非零列自上而下第一个非零元为 1（“主 1”），
- 主 1 所在行中其余元素（$n-1$ 个）均为 0。

# 列简化梯形矩阵（续）

$$A \underbrace{\xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}}}_{\text{对应初等矩阵}Q_j(1\le j\le s)} B = AQ_1Q_2\cdots Q_s$$ 列简化梯形矩阵

$B$ 有 $r$ 个非零列，即 $B$ 有 $r$ 个“主 1”，$(i_1,1),(i_2,2),\cdots,(i_r,r)$,

$$1 \le i_1 < i_2 < \cdots < i_r \le m \quad i_1 = 1, i_2 = 2, \cdots, i_r = r$$

$$B = \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & e_1^T \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & e_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & e_r^T \\ b_{r+1,1} & b_{r+1,2} & \cdots & b_{r+1,r} & 0 & \cdots & 0 & \beta_{r+1}^T \\ b_{r+2,1} & b_{r+2,2} & \cdots & b_{r+2,r} & 0 & \cdots & 0 & \beta_{r+2}^T \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{m1} & b_{m2} & \cdots & b_{mr} & 0 & \cdots & 0 & \beta_m^T \end{array}\right) \quad \beta_i = \begin{pmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{ir} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

# 列简化梯形矩阵（续）

$$A \underbrace{\xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}}}_{\text{对应初等矩阵}Q_j(1\le j\le s)} B = AQ_1Q_2\cdots Q_s \quad \text{列简化梯形矩阵}$$

$B$ 有 $r$ 个 "主 1"， $(i_1, 1), (i_2, 2), \cdots, (i_r, r)$,

$$1 \le i_1 < i_2 < \cdots < i_r \le m \quad i_1 = 1, i_2 = 2, \cdots, i_r = r$$

$$\begin{aligned}\beta_i^T = (b_{i1},\ b_{i2},\ \cdots,\ b_{ir},\ \overbrace{0,\ \cdots,\ 0}^{\text{位于零列}}) &= b_{i1}e_1 + b_{i2}e_2 + \cdots + b_{ir}e_r \\ &= b_{i1}\beta_{i_1} + b_{i2}\beta_{i_2} + \cdots + b_{ir}\beta_{i_r}\end{aligned}$$

- 有限次列变换必定可以将 $A$ 变换为"列简化梯形矩阵"；换言之，$A$ 必定与某个列简化梯形矩阵列等价

$$A \xrightarrow{c} \text{列简化梯形矩阵}$$

# 标准型矩阵

有限次初等变换，能将矩阵 $A=(a_{ij})_{m\times n}$ 简化到什么程度？

$$\underbrace{A \xrightarrow{\text{初等变换}} \cdots \xrightarrow{\text{初等变换}}}_{\text{对应初等矩阵}P_i,Q_j} B = P_k\cdots P_2P_1AQ_1Q_2\cdots Q_s \quad \overbrace{\underbrace{\text{标准型矩阵}}_{\text{列简化梯形}}}^{\text{行简化梯形}}$$

$$B=\left(\begin{array}{ccccccc|c}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & e_1^T\\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & e_2^T\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & e_r^T\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\
\hline
e_1 & e_2 & \cdots & e_r & 0 & \cdots & 0 &
\end{array}\right)=\begin{pmatrix} E_r & 0\\ 0 & 0\end{pmatrix}$$

$e_i$ $m-$维

$e_j$ $n-$维

$A\longrightarrow$ 标准形

# 矩阵的秩（rank）

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$1 \le i_1 < i_2 < \cdots < i_k \le m$$
$$1 \le j_1 < j_2 < \cdots < j_r \le n$$

$$\overbrace{\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_k,j_1} & a_{i_k,j_2} & \cdots & a_{i_k,j_r} \end{pmatrix}}^{\text{子矩阵}}$$

$k$阶子矩阵的行列式： $k$阶子式, $k = r$
$k$阶子式有： $C_m^k C_n^k$个
主子矩阵的行列式： 主子式

$k$阶主子矩阵 $\longrightarrow$
$k = r,\ \ i_1 = j_1, i_2 = j_2, \cdots, i_k = j_k$
$k$阶顺序主子矩阵： $i_1 = 1, \cdots, i_k = k$

$$\begin{pmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_k,i_1} & a_{i_k,i_2} & \cdots & a_{i_k,i_k} \end{pmatrix}$$

# 矩阵的秩（续）

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

矩阵的秩
$rank(A) = r(A)$
非负整数， $\leq \min\{m, n\}$

- $A = 0_{m\times n}$， $rank(A) = r(A) = 0$；
- 如果 $A$ 的所有 $r+1$ 阶子式均为 0，而至少有一个 $r$ 阶子式非 0，那么 $rank(A) = r(A) = r$；

## 矩阵秩的直观性质

- $0 \leq r(A) = r \leq \min\{m, n\}$，所有 $k(> r)$ 阶子式均为 0；
- 如果 $A$ 为 $n$ 阶可逆矩阵，则 $r(A) = n$，可逆又称“满秩”；
- $r(A) = r(A^T)$。

# 初等变换不改变秩 ($A \xrightarrow{\lambda r_i} B$，$r = r(A) = r(B)$)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\underbrace{\begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix}}_{=D\neq 0}$$

$$B = \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \lambda a_{i1} & \lambda a_{i2} & \cdots & \lambda a_{in} \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

$$\underbrace{\begin{vmatrix} b_{i_1,j_1} & b_{i_1,j_2} & \cdots & b_{i_1,j_r} \\ b_{i_2,j_1} & b_{i_2,j_2} & \cdots & b_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i_r,j_1} & b_{i_r,j_2} & \cdots & b_{i_r,j_r} \end{vmatrix}}_{=D'\neq 0}$$

$i = i_k \;\Rightarrow\; D' = \lambda D \neq 0$

$i \notin \{i_1, i_2, \cdots, i_r\} \;\Rightarrow\; D' = D \neq 0$

$B$有一个$k$阶子式非 0，$r(B) \geq r(A)$

$B \xrightarrow{1/\lambda r_i} A \quad \Rightarrow \quad r(A) \geq r(B)$

# 初等变换不改变秩（$A \xrightarrow{r_i \leftrightarrow r_k} B$，$r = r(A) = r(B)$）

$$A = \left(\begin{array}{cccc|c} \vdots & \vdots & \cdots & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} & i^{th} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kn} & k^{th} \\ \vdots & \vdots & \cdots & \vdots & \end{array}\right)$$

$$\underbrace{\begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix}}_{=D\neq 0}$$

$$B = \left(\begin{array}{cccc|c} \vdots & \vdots & \cdots & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kn} & i^{th} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} & k^{th} \\ \vdots & \vdots & \cdots & \vdots & \end{array}\right)$$

$$\underbrace{\begin{vmatrix} b_{i_1,j_1} & b_{i_1,j_2} & \cdots & b_{i_1,j_r} \\ b_{i_2,j_1} & b_{i_2,j_2} & \cdots & b_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ b_{i_r,j_1} & b_{i_r,j_2} & \cdots & b_{i_r,j_r} \end{vmatrix}}_{=D'\neq 0}$$

$$i,k \in \{i_1, i_2, \cdots, i_r\} \ \Rightarrow \ D' = -D$$
$$i,k \notin \{i_1, i_2, \cdots, i_r\} \ \Rightarrow \ D' = D$$

# 初等变换不改变秩（$A \xrightarrow{r_i \leftrightarrow r_k} B$，$r = r(A) = r(B)$，续）

$$B = \left(\begin{array}{cccc|c} \vdots & \vdots & \cdots & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kn} & i^{th} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} & k^{th} \\ \vdots & \vdots & \cdots & \vdots & \end{array}\right)$$

$i \in \{i_1, i_2, \cdots, i_r\}, k \notin \{i_1, i_2, \cdots, i_r\}$

取$B$的第 $\underbrace{i_1, \cdots, k, \cdots, i_r}_{\text{按照升序排列}}$ 行

仍旧取$B$的第$j_1, j_2, \cdots, j_r$列

$$\underbrace{\begin{vmatrix} \vdots & \vdots & \cdots & \vdots \\ b_{k,j_1} & b_{k,j_2} & \cdots & b_{k,j_r} \\ \vdots & \vdots & \cdots & \vdots \end{vmatrix}}_{=D'=\pm D\neq 0}$$

- $B$ 有 $k$ 阶子式非 0，$r(B) \geq r(A)$
-
$$A \xrightarrow{r_i \leftrightarrow r_k} B \xrightarrow{r_i \leftrightarrow r_k} A \quad \Rightarrow \quad r(A) \leq r(B)$$

# 初等变换不改变秩（$A \xrightarrow{r_i+\lambda r_k} B$，$r = r(A) = r(B)$）

$$A = \left(\begin{array}{cccc|c} \vdots & \vdots & \cdots & \vdots & \\ a_{i1} & a_{i2} & \cdots & a_{in} & i^{th} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{k1} & a_{k2} & \cdots & a_{kn} & k^{th} \\ \vdots & \vdots & \cdots & \vdots & \end{array}\right) \quad \overbrace{\begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix}}^{=D\neq 0}$$

$$B = \left(\begin{array}{ccc|c} \vdots & \vdots & \vdots & \\ a_{i1}+\lambda a_{k1} & \cdots & a_{in}+\lambda a_{kn} & i^{th} \\ \vdots & \vdots & \vdots & \end{array}\right)$$

$i \notin \{i_1, i_2, \cdots, i_r\}$
取$B$的$i_1, \cdots, i_r$行
取$B$的$j_1, \cdots, j_r$列
$\Rightarrow \quad D' = D \neq 0$

$$D' = \begin{vmatrix} \vdots & \cdots & \vdots & \\ a_{i,j_1}+\lambda a_{k,j_1} & \cdots & a_{i,j_r}+\lambda a_{k,j_r} & i^{th} \\ \vdots & \cdots & \vdots & \end{vmatrix}$$

$i, k \in \{i_1, \cdots, i_r\}$
$D \xrightarrow{r_i+\lambda r_k} D, D' = D$

# 初等变换不改变秩（$A \xrightarrow{r_i+\lambda r_k} B$，$r=r(A)=r(B)$，续）

$$B=\left(\begin{array}{ccc|c} \vdots & \vdots & \vdots & \\ a_{i1}+\lambda a_{k1} & \cdots & a_{in}+\lambda a_{kn} & i^{th} \\ \vdots & \vdots & \vdots & \end{array}\right)$$

$i\in\{i_1,i_2,\cdots,i_r\}$
$k\notin\{i_1,i_2,\cdots,i_r\}$
$D'=D+\lambda D_1$

$$D_1=\left|\begin{array}{cccc} \vdots & \cdots & \vdots & \\ a_{k,j_1} & \cdots & a_{k,j_r} & i^{th} \\ \vdots & \cdots & \vdots & \end{array}\right|$$

$D_1=0$，取$B$的$i_1,\cdots,i_r$行，
$D'=D\neq 0$
$D_1\neq 0$，取$B$的$k$行代替$i$行，
$D'=\pm D_1\neq 0$

- $B$ 有 $k$ 阶子式非 0，$r(B)\geq r(A)$
- $$A \xrightarrow{r_i+\lambda r_k} B \xrightarrow{r_i-\lambda r_k} A \quad\Rightarrow\quad r(A)\leq r(B)$$

# 初等变换不改变秩（续，行变换计算秩）

- $A \xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}} B$，则 $r(A)=r(B)$；$B$ 行简化梯形？

$B$ 有 $r$ 个非零行，即 $B$ 有 $r$ 个“主 1”，$(1,j_1),(2,j_2),\cdots,(r,j_r)$

$1 \le j_1 < j_2 < \cdots < j_r \le n$  $j_1=1, j_2=2, \cdots, j_r=r$

$$B=\left(\begin{array}{cccccccc} 1 & 0 & \cdots & 0 & b_{1,r+1} & b_{1,r+2} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{2,r+1} & b_{2,r+2} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{r,r+1} & b_{r,r+2} & \cdots & b_{rn} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline e_1 & e_2 & \cdots & e_r & \beta_{r+1} & \beta_{r+2} & \cdots & \beta_n \end{array}\right)$$

取$1,2,\cdots,r$行
取$j_1,j_2,\cdots,j_r$列
$D=|E_r|=1$
$r+1$阶子式
含零行，必为 0

$r(B)=r$

# 初等变换不改变秩（续，行变换计算秩）

- $A \xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}} B$，则 $r(A)=r(B)$；$B$ 列简化梯形？

$B$ 有 $r$ 个非零列，即 $B$ 有 $r$ 个 “主 1”，$(i_1,1),(i_2,2),\cdots,(i_r,r)$,

$1 \le i_1 < i_2 < \cdots < i_r \le m \quad i_1=1, i_2=2, \cdots, i_r=r$

$$B = \left( \begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & e_1^T \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & e_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & e_r^T \\ b_{r+1,1} & b_{r+1,2} & \cdots & b_{r+1,r} & 0 & \cdots & 0 & \beta_{r+1}^T \\ b_{r+2,1} & b_{r+2,2} & \cdots & b_{r+2,r} & 0 & \cdots & 0 & \beta_{r+2}^T \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ b_{m1} & b_{m2} & \cdots & b_{mr} & 0 & \cdots & 0 & \beta_m^T \end{array} \right)$$

取$1,2,\cdots,r$列
取$i_1,i_2,\cdots,i_r$行
$D=|E_r|=1$
$r+1$阶子式
含零列，必为 0

$r(B)=r$

# 初等变换不改变秩（续，行变换计算秩）

- $A \xrightarrow{\text{初等变换}} \cdots \xrightarrow{\text{初等变换}} B$，则 $r(A) = r(B)$；$B$ 标准型矩阵？

$$B = \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & e_1^T \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & e_2^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & e_r^T \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \\ \hline e_1 & e_2 & \cdots & e_r & 0 & \cdots & 0 & \end{array}\right) = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$$

取$1, 2, \cdots, r$行
取$1, 2, \cdots, r$列
$D = |E_r| = 1$
$r+1$阶子式为 0

$r(B) = r$

所有 $m \times n$ 实（复）矩阵的集合记为 $R^{m\times n}$（$C^{m\times n}$），按照等价关系 $\longrightarrow$ 划分为 $1 + \min\{m, n\}$ 类，每一类矩阵彼此等价，有相同的标准型矩阵。

# 初等变换不改变秩（续，逆矩阵计算）

$A = (a_{ij})_{n\times n}$ 可逆，$r(A) = n$

$$A \underbrace{\xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵} P_i (1\le i\le k)} B = P_k \cdots P_2 P_1 A \quad \text{行简化梯形矩阵}$$

$B = (b_{ij})_{n\times n}$ 有 $n$ 个 "主 1"，$(1, j_1), (2, j_2), \cdots, (n, j_n)$

$$1 \le j_1 < j_2 < \cdots < j_n \le n \quad \Rightarrow \quad j_1 = 1, j_2 = 2, \cdots, j_n = n$$

逆矩阵的计算方法（编程实现）

$$\left.\begin{array}{c} E = B = P_k \cdots P_2 P_1 A \\ \Downarrow \\ P_k \cdots P_2 P_1 E = A^{-1} \\ \underbrace{P_k \cdots P_2 P_1 C = A^{-1} C}_{C=(c_{ij})_{n\times k}} \end{array}\right\} \Rightarrow \begin{array}{l} (A\ E) \xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}} (E\ A^{-1}) \\ (A\ C) \xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}} (E\ A^{-1}C) \end{array}$$

# 初等变换不改变秩（续，逆矩阵计算）

$A=(a_{ij})_{n\times n}$ 可逆，$r(A)=n$

$$A \underbrace{\xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}}}_{\text{对应初等矩阵}Q_j(1\le j\le s)} B=AQ_1Q_2\cdots Q_s \quad \text{列简化梯形矩阵}$$

$B=(b_{ij})_{n\times n}$ 有 $n$ 个 “主 1”，$(i_1,1),(i_2,2),\cdots,(i_n,n)$

$$1\le i_1<i_2<\cdots<i_n\le n \quad \Rightarrow \quad i_1=1,i_2=2,\cdots,i_n=n$$

逆矩阵的计算方法（编程实现）

$$\left.\begin{array}{c} E=B=AQ_1Q_2\cdots Q_s \\ \Downarrow \\ EQ_1Q_2\cdots Q_s=A^{-1} \\ \underbrace{CQ_1Q_2\cdots Q_s=CA^{-1}}_{C=(c_{ij})_{k\times m}} \end{array}\right\} \Rightarrow \begin{array}{c} \begin{pmatrix} A \\ E \end{pmatrix} \xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}} \begin{pmatrix} E \\ A^{-1} \end{pmatrix} \\ \begin{pmatrix} A \\ C \end{pmatrix} \xrightarrow{\text{列变换}} \cdots \xrightarrow{\text{列变换}} \begin{pmatrix} E \\ CA^{-1} \end{pmatrix} \end{array}$$

# 向量组的线性组合与线性方程组

$$A=\left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & a_{22} & \cdots & a_{2n} \\ \hline \vdots & \vdots & \cdots & \vdots \\ \hline a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right)\begin{array}{c}\beta_1^T\\ \beta_2^T\\ \vdots\\ \beta_m^T\end{array} \qquad x=\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} \quad y=\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_m\end{pmatrix}$$

$$\alpha_1 \qquad \alpha_2 \qquad \cdots \qquad \alpha_n$$

$$Ax = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

$$y^T A = y_1\beta_1^T + y_2\beta_2^T + \cdots + y_m\beta_m^T$$

$\uparrow$ $x_j, y_i$称为乘子

向量组的线性组合

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\ \cdots\ \cdots\ \cdots\cdots\ \cdots\ \cdots\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ \hline x_1\cdot\alpha_1 \qquad x_2\cdot\alpha_2 \qquad\qquad x_n\cdot\alpha_n \qquad b \end{array}\right.$$

$$x=(x_1, x_2, \cdots, x_n)^T$$

$$A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)$$

$$\underbrace{x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n}_{=Ax}$$

$Ax = 0/b \neq 0$ 齐次/非齐次

# 向量组的线性相关与表出

$b$ 可由向量组 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性表出：若有 $c_1, c_2, \cdots, c_n$，

$$b = \sum_{j=1}^{n} c_j \alpha_j = \underbrace{\overbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n}^{=Ac,\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n)}}_{\text{线性方程组}Ax=b\text{有解}c}, \quad c = (c_1, c_2, \cdots, c_n)^T$$

$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性相关：若有**不全为 0 的数** $c_1, c_2, \cdots, c_n$，

$$\sum_{j=1}^{n} c_j \alpha_j = \underbrace{\overbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n}^{=Ac,\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n)} = 0}_{\text{齐次线性方程组}Ax=0\text{有非零解}c}, \quad \underbrace{c = (c_1, c_2, \cdots, c_n)^T \neq 0}_{c_1,c_2,\cdots,c_n\text{不全为 0}}$$

$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性无关（线性不相关、线性独立）：

$$\underbrace{\overbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n}^{=Ac,\ A=(\alpha_1,\alpha_2,\cdots,\alpha_n)} = 0 \implies c_1 = c_2 = \cdots = c_n = 0\ (c = 0)}_{\text{齐次线性方程组}Ax=0\text{只有零解}}$$

# 向量组的线性相关与表出（续）

- 向量组 $\{\beta_1, \beta_2, \cdots, \beta_m\}$ 可由向量组 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 表出：

$$\beta_j = c_{1j}\alpha_1 + c_{2j}\alpha_2 + \cdots + c_{nj}\alpha_n \quad (1 \leq j \leq m)$$

$$\underbrace{(\beta_1 \ \beta_2 \ \cdots \ \beta_m)}_{=(b_{ij})_{k\times m}=B} = \underbrace{(\alpha_1 \ \alpha_2 \ \cdots \alpha_n)}_{=(a_{ij})_{k\times n}=A} \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nm} \end{pmatrix}}_{=(c_{ij})_{n\times m}=C}$$

$B = AC$，等价于，$B$ 的列向量组可由 $A$ 的列向量组表出

- 如果 $\{\beta_1, \beta_2, \cdots, \beta_m\}$ 与 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 可以相互线性表出，则称这两个向量组等价，存在矩阵 $C$ 与 $D$ 使得：

$$B = AC, \quad A = BD$$

# 向量组的线性相关与表出（续，性质）

- 零向量可由向量组 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 表出（$Ax = 0$ 必有解）
- $\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$，则 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性相关

$$\underbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + (-1)\beta = 0}_{c_1,\ c_2,\ \cdots,\ c_n,\ -1\ \text{不全为零}}$$

- $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性无关，$\{\alpha_1, \alpha_2, \cdots, \alpha_n, \beta\}$ 线性相关，则 $\beta$ 可由向量组 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 表出

$$\underbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n + c\beta = 0}_{c_1,\ c_2,\ \cdots,\ c_n,\ c\ \text{不全为零}}$$

$$c = 0 \times \quad \Rightarrow \quad \overbrace{c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n}^{c_1,\ c_2,\ \cdots,\ c_n,\ \text{不全为零}} + \overbrace{c\beta}^{=0} = 0 \quad \text{矛盾!}$$

$$c \neq 0 \checkmark \quad \Rightarrow \quad \beta = (-\frac{c_1}{c})\alpha_1 + (-\frac{c_2}{c})c_2\alpha_2 + \cdots + (-\frac{c_n}{c})\alpha_n$$

# 向量组的线性相关与表出（续，性质）

- $\beta$ 可由 $\mathcal{S} = \{\overbrace{\alpha_1, \alpha_2, \cdots, \alpha_n}^{\text{线性无关}}\}$ 表出，则表出方式唯一

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_n\alpha_n$$

$$(c_1 - d_1)\alpha_1 + (c_2 - d2)\alpha_2 + \cdots + (c_n - d_n)\alpha_n = 0$$

$$\Rightarrow \quad c_i - d_i = 0 \ (1 \le i \le n) \quad \text{线性表出方式唯一}$$

- 部分组 $\mathcal{T}$ 线性相关，则 $\mathcal{S}$ 也线性无关

$$\mathcal{T} = \{\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}\} \subseteq \{\alpha_1, \alpha_2, \cdots, \alpha_n\} = \mathcal{S}$$

$$\begin{aligned} 0 &= \overbrace{c_{j_1}\alpha_{j_1} + c_{j_2}\alpha_{j_2} + \cdots + c_{j_r}\alpha_{j_r}}^{c_{j_1}, c_{j_2}, \cdots, c_{j_r}\text{不全为零}} \quad \text{“小”的向量组线性相关} \\ &= \underbrace{c_{j_1}\alpha_{j_1} + c_{j_2}\alpha_{j_2} + \cdots + c_{j_r}\alpha_{j_r} + \sum_{j \notin \{j_1, \cdots, j_r\}} 0 \cdot \alpha_j}_{c_{j_1}, c_{j_2}, \cdots, c_{j_r}, 0, \cdots, 0\text{不全为零，“大”的向量组也线性相关}} \end{aligned}$$

# 向量组的线性相关与表出（续，性质）

- 向量组 $\{0\}$ 线性相关；包含 0 向量的向量组线性相关；
- 向量组 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性相关当且仅当：有一个 $\alpha_j$ 可由其余 $n-1$ 个向量线性表出

$$\begin{aligned} \alpha_j &= c_1\alpha_1 + \cdots + c_{j-1}\alpha_{j-1} + c_{j+1}\alpha_{j+1} + \cdots + +c_n\alpha_n \\ \Rightarrow \quad 0 &= \underbrace{c_1\alpha_1 + \cdots + c_{j-1}\alpha_{j-1} + 1 \cdot \alpha_j + c_{j+1}\alpha_{j+1} + \cdots + c_n\alpha_n}_{c_1,\cdots,c_{j-1},c_j=1,\cdot,c_{j+1},\cdots,c_n \text{ 不全为 0 } \Rightarrow \ \alpha_1,\alpha_2,\cdots,\alpha_n \text{ 线性相关}} \end{aligned}$$

$$\begin{aligned} 0 &= \overbrace{c_1\alpha_1 + \cdots + c_{j-1}\alpha_{j-1} + c_j\alpha_j + c_{j+1}\alpha_{j+1} + \cdots + c_n\alpha_n}^{\alpha_1,\alpha_2,\cdots,\alpha_n \text{ 线性相关 } \Rightarrow \ c_1,\cdots,c_{j-1},c_j\neq 0,\cdot,c_{j+1},\cdots,c_n \text{ 不全为 0}} \\ -c_j\alpha_j &= c_1\alpha_1 + \cdots + c_{j-1}\alpha_{j-1} + c_{j+1}\alpha_{j+1} + \cdots + c_n\alpha_n \quad \Rightarrow \end{aligned}$$

$$\alpha_j = -\frac{c_1}{c_j}\alpha_1 - \cdots - \frac{c_{j-1}}{c_j}\alpha_{j-1} - \frac{c_{j+1}}{c_j}\alpha_{j+1} - \cdots - \frac{c_n}{c_j}\alpha_n$$

# 初等变换判断线性相关性

$$A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)\ \underbrace{\xrightarrow{\text{行变换}}\ \cdots\ \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)}\ \underbrace{(\alpha_1'\ \alpha_2'\ \cdots\ \alpha_n')=A'=PA}_{P=P_k\cdots P_2P_1\ \text{可逆}}$$

$$PA=P(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)=\underbrace{(P\alpha_1\ P\alpha_2\ \cdots\ P\alpha_n)=(\alpha_1'\ \alpha_2'\ \cdots\ \alpha_n')}_{\alpha_j'=P\alpha_j,\ \alpha_j=P^{-1}\alpha_j',\ 1\le j\le n}=A'$$

$$P\overbrace{(c_1\alpha_1+c_2\alpha_2+\cdots+c_n\alpha_n)}^{=\beta}=\overbrace{c_1\underbrace{P\alpha_1}_{\alpha_1'}+c_2\underbrace{P\alpha_2}_{\alpha_2'}+\cdots+c_n\underbrace{P\alpha_n}_{\alpha_n'}}^{=\beta'=P\beta}$$

- $A\xrightarrow{r}A'$，$A$ 与 $A'$ 的列向量组同线性相关、同线性无关！

$$\beta=0\ \Rightarrow\ \beta'=P\beta=0,\qquad \beta'=0\ \Rightarrow\ \beta=P^{-1}\beta'=0$$

# 初等变换判断线性相关性（续）

$$A = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n) \xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}} \underbrace{(\beta_1\ \beta_2\ \cdots\ \beta_n) = B = PA}_{B\text{行简化梯形，}P\text{可逆}}$$

$B$ 有 $r = r(A) = r(A')$ 个“主 1”，$(1, j_1), (2, j_2), \cdots, (r, j_r)$

$1 \le j_1 < j_2 < \cdots < j_r \le n \qquad r < n \ \Rightarrow\ \exists j \notin \{j_1, j_2, \cdots, j_r\}$

$$B = \left(\begin{array}{ccccccc} 1 & 0 & \cdots & 0 & b_{1,r+1} & \cdots & b_{1n} \\ 0 & 1 & \cdots & 0 & b_{2,r+1} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{r,r+1} & \cdots & b_{rn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline e_1 & e_2 & \cdots & e_r & \beta_{r+1} & \cdots & \beta_n \end{array}\right)$$

$$\beta_j = \begin{pmatrix} b_{1j} \\ \vdots \\ b_{rj} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad e_i = \beta_{j_i}$$

$$= \sum_{i=1}^{r} b_{ij} e_i = \sum_{i=1}^{r} b_{ij} \beta_{j_i}$$

$\beta_j$可由$\beta_{j_1}, \cdots, \beta_{j_r}$表出

# 初等变换判断线性相关性（续）

- $r = r(A) = r(B) < n$，$\alpha_1, \alpha_2, \cdots, \alpha_n$ **线性相关**；

$$\exists j \notin \{j_1, j_2, \cdots, j_r\}, \quad \beta_j \text{可由} \beta_{j_1}, \cdots, \beta_{j_r} \text{表出}$$

$$\Rightarrow \{\beta_j, \beta_{j_1}, \cdots, \beta_{j_r}\}\text{线性相关} \Rightarrow \underbrace{\{\beta_1, \beta_2, \cdots, \beta_n\}\text{线性相关}}_{\text{“大”向量组}}$$

$$\Rightarrow \{\alpha_1, \alpha_2, \cdots, \alpha_n\}\text{线性相关} \quad \beta_j = \sum_{i=1}^{r} b_{ij}\beta_{j_i}, \quad \alpha_j = \sum_{i=1}^{r} b_{ij}\alpha_{j_i}$$

- $r = r(A) = r(B) = n$，$\alpha_1, \alpha_2, \cdots, \alpha_n$ **线性无关**；

$$B = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

$$1 \le j_1 < j_2 < \cdots < j_n \le n$$
$$\Rightarrow \quad j_1 = 1, j_2 = 2, \cdots, j_n = n$$
$$B = (e_1, e_2, \cdots, e_n), \quad e_j\text{是}m\text{维的}$$
$$\beta_1 = e_1, \beta_2 = e_2, \cdots, \beta_n = e_n\text{线性无关}$$

若$m < n$，则$\alpha_1, \cdots, \alpha_n$线性相关 $(r(A) \le m)$

# 初等变换判断线性相关性（续）

- $n$ 个 $n$ 维向量 $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ 线性无关，当且仅当

$$|A| = |(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)| \neq 0$$

- $A = (a_{ij})_{m\times n}$，$r(A) = r$，$A$ 有 $r$ 个线性无关的列/行向量。

$$1 \le i_1 < i_2 < \cdots < i_r \le m, \quad 1 \le j_1 < j_2 < \cdots < j_r \le m$$

$$0 \neq \begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \cdots & a_{i_1,j_r} \\ a_{i_2,j_1} & a_{i_2,j_2} & \cdots & a_{i_2,j_r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i_r,j_1} & a_{i_r,j_2} & \cdots & a_{i_r,j_r} \end{vmatrix} \quad \begin{matrix} A_1 = (\alpha_{j_1}\ \cdots\ \alpha_{j_r})\ r(A_1) = r \\ A_2 = \begin{pmatrix} \beta_{i_1} \\ \vdots \\ \beta_{i_r} \end{pmatrix}\ r(A_2) = r \end{matrix}$$

- $A \xrightarrow{r/c} B$，则 $A$ 行/列向量组与 $B$ 行/列向量组等价；反之？

$$\overbrace{A = \underbrace{P_k \cdots P_2 P_1}_{P=(p_{ij})\text{可逆}} B}^{B=P^{-1}A,\ P^{-1}=(q_{ij})}, \quad A(i,:) = \sum_j p_{ij} B(j,:),\ B(i,:) = \sum_j q_{ij} A(j,:)$$

# 极大线性无关组

$$\underbrace{\overbrace{\mathcal{T}=\{\alpha_{j_1},\alpha_{j_2},\cdots,\alpha_{j_r}\}}^{\text{部分组}}\subseteq\mathcal{S}=\{\alpha_1,\alpha_2,\cdots,\alpha_n\},\quad \overbrace{\alpha_{j_k}}^{\in\mathcal{T}}=1\cdot\overbrace{\alpha_{j_k}}^{\in\mathcal{S}}}_{\text{部分组（小）}\mathcal{T}\text{可由（大）向量组}\mathcal{S}\text{表出}}$$

部分组（小）$\mathcal{T}$可由（大）向量组$\mathcal{S}$表出，$\mathcal{S}$可由什么样的部分组表出？

$\mathcal{T}=\{\alpha_{j_1},\alpha_{j_2},\cdots,\alpha_{j_r}\}$ 是 $\mathcal{S}$ 的一个极大（线性）无关组，若

$$\underbrace{\overbrace{\alpha_{j_1},\alpha_{j_2},\cdots,\alpha_{j_r}}^{\text{线性无关}},\ \alpha_j}_{\text{线性相关}}\quad(\forall\alpha_j\in\ \mathcal{S}-\mathcal{T})$$

- $\mathcal{T}=\{\alpha_{j_1},\alpha_{j_2},\cdots,\alpha_{j_r}\}$ 是 $\mathcal{S}$ 的一个极大无关组，则任意 $\alpha_j$ 可由 $\mathcal{T}$ 唯一地表出，$\mathcal{S}$ 与 $\mathcal{T}$ 等价（相互表出）；
- 两个极大无关组等价；两个向量组 $\mathcal{S}_1$ 与 $\mathcal{S}_1$ 等价，当且仅当它们的极大无关组等价。

# 初等变换计算极大无关组

$$A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)\ \underbrace{\xrightarrow{\text{行变换}}\cdots\xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)}\ \underbrace{(\alpha_1'\ \alpha_2'\ \cdots\ \alpha_n')=A'=PA}_{P=P_k\cdots P_2P_1\ \text{可逆}}$$

$$PA=P(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)=\underbrace{(P\alpha_1\ P\alpha_2\ \cdots\ P\alpha_n)=(\alpha_1'\ \alpha_2'\ \cdots\ \alpha_n')}_{\alpha_j'=P\alpha_j,\ \alpha_j=P^{-1}\alpha_j',\ 1\le j\le n}=A'$$

$$P\overbrace{(c_1\alpha_{j_1}+c_2\alpha_{j_2}+\cdots+c_r\alpha_{j_r})}^{=\beta}=\overbrace{c_1\underbrace{P\alpha_{j_1}}_{\alpha_{j_1}'}+c_2\underbrace{P\alpha_{j_2}}_{\alpha_{j_2}'}+\cdots+c_r\underbrace{P\alpha_{j_r}}_{\alpha_{j_r}'}}^{=\beta'=P\beta}$$

$$\beta=0\ \Rightarrow\ \beta'=P\beta=0,\qquad \beta'=0\ \Rightarrow\ \beta=P^{-1}\beta'=0$$

$$\begin{matrix}\{\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n\}\ \supseteq\ \{\alpha_{j_1},\ \alpha_{j_2},\ \cdots,\ \alpha_{j_r}\}\\ \{\alpha_1',\ \alpha_2',\ \cdots,\ \alpha_n'\}\ \supseteq\ \{\alpha_{j_1}',\ \alpha_{j_2}',\ \cdots,\ \alpha_{j_r}'\}\end{matrix}\quad \text{同时线性无关/相关}$$

# 初等变换计算极大无关组（续）

$$A = (\overbrace{\alpha_1\ \ \alpha_2\ \ \cdots\ \ \alpha_n}^{\mathcal{S}}) \underbrace{\xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}}}_{\text{对应初等矩阵}P_i(1\le i\le k)} \underbrace{\overbrace{(\alpha_1'\ \ \alpha_2'\ \ \cdots\ \ \alpha_n')}^{\mathcal{S}'} = A' = PA}_{P=P_k\cdots P_2P_1\ \text{可逆}}$$

$$\underbrace{\overbrace{\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}}^{\text{线性无关}},\ \ \alpha_j}_{\text{线性相关}} \Leftrightarrow \underbrace{\overbrace{\alpha_{j_1}', \alpha_{j_2}', \cdots, \alpha_{j_r}'}^{\text{线性无关}},\ \ \alpha_j'}_{\text{线性相关}} \quad (\forall j \notin \{j_1, j_2, \cdots, j_r\})$$

$$\alpha_j = c_1\alpha_{j_1} + c_2\alpha_{j_2} + \cdots c_r\alpha_{j_r} \Leftrightarrow \alpha_j' = c_1\alpha_{j_1}' + c_2\alpha_{j_2}' + \cdots + c_r\alpha_{j_r}'$$

- $\{\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}\}$ 是 $\mathcal{S}$ 的一个极大无关组，当且仅当，$\{\alpha_{j_1}', \alpha_{j_2}', \cdots, \alpha_{j_r}'\}$ 是 $\mathcal{S}'$ 的一个极大无关组；
- $\alpha_j \in \mathcal{S}$ 由 $\{\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}\}$ 线性表出的方式与 $\alpha_j' \in \mathcal{S}'$ 由 $\{\alpha_{j_1}', \alpha_{j_2}', \cdots, \alpha_{j_r}'\}$ 线性表出的方式相同。

# 初等变换计算极大无关组（续）

$$A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)\xrightarrow{\text{行变换}}\cdots\xrightarrow{\text{行变换}}\underbrace{(\beta_1\ \beta_2\ \cdots\ \beta_n)=B=PA}_{B\text{行简化梯形，}P\text{可逆}}$$

$B$ 有 $r=r(A)=r(A')$ 个 “主 1”，$(1,j_1),(2,j_2),\cdots,(r,j_r)$

$1\le j_1<j_2<\cdots<j_r\le n$ $\quad r<n\ \Rightarrow\ \exists j\notin\{j_1,j_2,\cdots,j_r\}$

$$B=\left(\begin{array}{ccccccc}1&0&\cdots&0&b_{1,r+1}&\cdots&b_{1n}\\0&1&\cdots&0&b_{2,r+1}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots&\vdots&\cdots&\vdots\\0&0&\cdots&1&b_{r,r+1}&\cdots&b_{rn}\\0&0&\cdots&0&0&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\cdots&\vdots\\0&0&\cdots&0&0&\cdots&0\\\hline e_1&e_2&\cdots&e_r&\beta_{r+1}&\cdots&\beta_n\end{array}\right)\qquad \beta_j=\begin{pmatrix}b_{1j}\\\vdots\\b_{rj}\\0\\\vdots\\0\end{pmatrix}$$

$$e_i=\beta_{j_i}$$
$$=\sum_{i=1}^r b_{ij}e_i$$
$$=\sum_{i=1}^r b_{ij}\beta_{j_i}$$

$\beta_j$可由$\beta_{j_1},\cdots,\beta_{j_r}$表出

# 初等变换计算极大无关组（续）

$$A = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n) \xrightarrow{\text{行变换}} \cdots \xrightarrow{\text{行变换}} \underbrace{(\beta_1\ \beta_2\ \cdots\ \beta_n) = B = PA}_{B\text{行简化梯形，}P\text{可逆}}$$

$B$ 有 $r = r(A) = r(A')$ 个"主 1"，$(1, j_1), (2, j_2), \cdots, (r, j_r)$

$$1 \leq j_1 < j_2 < \cdots < j_r \leq n \qquad r < n \ \Rightarrow\ \exists j \notin \{j_1, j_2, \cdots, j_r\}$$

$$\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \supseteq \{\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}\} \text{ 极大无关组}$$
$$\{\beta_1, \beta_2, \cdots, \beta_n\} \supseteq \{\beta_{j_1}, \beta_{j_2}, \cdots, \beta_{j_r}\} \text{ 极大无关组}$$

$$\beta_j = (b_{1j}, \cdots, b_{rj}, \overbrace{0, \cdots, 0}^{\text{位于零行}})^T = \sum_{i=1}^{r} b_{ij} e_i = \sum_{i=1}^{r} b_{ij} \beta_{j_i} \qquad \beta_{j_i} = e_i$$

- 极大无关组中含有 $r = r(A) = r(B)$ 个向量！不同极大无关组含有相同数量的向量？

# 不同极大无关组含有相同数量的向量？

- $\{\beta_1,\cdots,\beta_n\}$ 线性无关，可由 $\{\alpha_1,\cdots,\alpha_m\}$ 表出，则 $m \ge n$；

$$\overbrace{\underbrace{B=(\beta_1\ \beta_2\ \cdots\ \beta_n)}_{B=AC,\ C=(c_{ij})_{m\times n}}}^{\text{线性无关，}r(B)=?} = \overbrace{\underbrace{(\alpha_1\ \alpha_2\ \cdots\ \alpha_m)}_{m<n\,?}}^{A} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$$

$$C \longrightarrow C' = \begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix} = PCQ,\ P=(p_{ij})_{m\times m},\ Q=(q_{ij})_{n\times n}\ \text{可逆}$$

$$0 \ne y^* = (\overbrace{0,\cdots,0}^{r\text{个}},1,0,\cdots,0)^T,\quad 0 = C'y^* = PC\overbrace{Qy^*}^{x^*\ne 0} \Rightarrow Cx^* = 0$$

$$\underbrace{x_1^*\beta_1 + x_2^*\beta_2 + \cdots + x_m^*\beta_m = Bx^* = \overbrace{AC}^{=B}x^* = A\overbrace{Cx^*}^{=0} = 0,\quad x^* \ne 0}_{x_1^*\beta_1+x_2^*\beta_2+\cdots+x_m^*\beta_m=0,\ x_1^*,x_2^*,\cdots,x_m^*\text{不全为零，}\beta_1,\beta_2,\cdots,\beta_m\ \text{线性相关，矛盾！}}$$

# 不同极大无关组含有相同数量的向量？(续)

- $\{\beta_1,\cdots,\beta_n\}$ 线性无关，可由 $\{\alpha_1,\cdots,\alpha_m\}$ 表出，则 $m\geq n$；
- $\{\beta_1,\cdots,\beta_n\}$ 与 $\{\alpha_1,\cdots,\alpha_m\}$ 线性无关且等价，则 $m=n$；
- $\{\alpha_1,\cdots,\alpha_m\}$ 的两个极大无关组含有相同数量的向量；

**向量组 $\{\alpha_1,\cdots,\alpha_m\}$ 的秩**：极大无关组包含向量的个数

$$\underbrace{\overbrace{r\{\alpha_1,\cdots,\alpha_m\}}^{\text{向量组的秩}}=r(A)\ \text{（矩阵的列秩）},\quad A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_m)}_{r(A)=r(A^T)=A^T\text{的列秩}=A\text{的行向量组的秩（矩阵的行秩）}}$$

- $\mathcal{S}=\{\beta_1,\cdots,\beta_n\}$ 可由 $\mathcal{T}=\{\alpha_1,\cdots,\alpha_m\}$ 表出，$r(\mathcal{S})\leq r(\mathcal{T})$；
$\mathcal{S}$ 的极大无关组可由 $\mathcal{T}$ 的极大无关组表出。

$$\underbrace{\overbrace{(\beta_1\ \beta_2\ \cdots\ \beta_n)}^{=B}=\overbrace{(\alpha_1\ \alpha_2\ \cdots\ \alpha_m)}^{=A}C_{m\times n}}_{\{\beta_1,\cdots,\beta_n\}\text{可由}\{\alpha_1,\cdots,\alpha_m\}\text{表出}}\Rightarrow$$

$$r(B)\leq r(A)$$

$$B^T=C^TA^T\Rightarrow$$
$$r(B)\leq r(C)$$

# 不同极大无关组含有相同数量的向量？(续)

- $r(A+B) \le r(A) + r(B)$；

$$\overbrace{A+B = (\underbrace{\alpha_1+\beta_1}_{\gamma_1} \ \ \underbrace{\alpha_2+\beta_2}_{\gamma_2} \ \cdots \ \underbrace{\alpha_m+\beta_m}_{\gamma_m})}^{A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_m),\ B=(\beta_1,\ \beta_2\ \cdots\ \beta_m)}$$

$\gamma_1, \cdots, \gamma_n$ 可由 $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ 表出 $\Rightarrow$ ？

- $C = A_{m\times n}B_{n\times k}$，$r(AB) \ge r(A) + r(B) - n$；

$$\begin{pmatrix} E & 0 \\ -A & E \end{pmatrix}\begin{pmatrix} B & E \\ 0 & A \end{pmatrix} = \underbrace{\begin{pmatrix} B & E \\ -AB & 0 \end{pmatrix}}_{r=n+r(AB)}$$

$AB = 0 \Rightarrow r(A) + r(B) \le n$

$$r(A) + r(B) = r\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \le r\begin{pmatrix} B & E \\ 0 & A \end{pmatrix}$$

$|A| = 0,\ AA^* = 0$
$r(A) + r(A^*) \le n$

- $A = (a_{ij})_{n\times n}$；$r(A) = n$，则 $r(A^*) = n$；$r(A) < n-1$，则 $A^* = 0$；$r(A) = n-1$（$A$ 有非零余子式），则 $r(A^*) = 1$。